

Introduction to Quantum Optics: an amateur's view

Lecture notes

M.I. Petrov, D.F. Kornovan, I.V. Toftul

ITMO University, Department of Physics and Mathematics

Autumn, 2019

Preface

I am very grateful to Andrey Bogdanov who helped me to organize this course, and to Kristina Frizyuk for her enormous help in preparing this manuscript.

Contents

Recommended literature	5
1 Atom-field interaction. Semiclassical theory	6
Homework	10
2 Density matrix of two energy level system	12
2.1 Density matrix of a subsystem	13
2.2 Density matrix of a mixed state	14
2.3 Density matrix of a two-level system	15
2.4 Bloch sphere	16
2.5 Dissipations	17
2.5.1 Spontaneous emission of TLS	17
2.6 Dielectric constant of media	18
2.7 Homework	20
Homework	20
3 Secondary quantization	21
3.1 Vector potential of the electromagnetic field	21
3.2 Field in the box, harmonics expansion, and the energy of the electromag-	
netic field	22
3.3 Field quantization	24
3.4 Ladder operators. Fock state. Second quantization	25
3.5 Fields' fluctuation	26
3.6 Homework	27
4 Coherent states	28
4.1 Eigenstates of annihilation operator	28
4.2 Basic properties of coherent states	29
Homework	30
4.3 Classical field	30
Homework	31
4.4 Fluctuations	31
4.5 Squeezed states or getting the maximum accuracy!	34
Homework	35
5 The coherence of light	37
5.1 Michelson stellar interferometer	37
5.2 Quantum theory of photodetection	40
6 Atom-field interaction. Quantum approach	42
6.1 Jaynes-Cummings model (RWA)	42
Homework	46
6.2 Collapse and revival	46
6.3 Energy spectrum. Dispersion relation	46
7 Spontaneous relaxation. Weisskopf-Wigner theory	51

8	Dipole radiation. Dyadic Green's function. The Purcell effect: classical approach	55
8.1	Dipole radiation and dyadic Green's function	55
8.1.1	Derivation of the Green's function for Maxwell equations	56
8.1.2	Near-, intermediate- and far-field parts of Green's function	57
8.2	Spontaneous relaxation and local density-of-state (IN A MIXED UNITS) .	57
8.2.1	An expression for spontaneous decay	57
8.2.2	Spontaneous decay and Green's dyadics	59
8.3	The Purcell factor	60
	Homework	61
9	Theory of relaxation of electromagnetic field. Heisenberg–Langevin method	62
9.1	In previous series	62
9.2	the Heisenberg–Langevin equation	62
	Homework	64
9.3	Properties of the stochastic operator	64
9.4	Equation of motion for the field correlation functions. Wiener–Khinchine theorem	65
10	Atom in a damped cavity	68
10.1	The Purcell factor for a closed cavity	68
10.2	Rigorous derivation of the atomic decay	69
11	Casimir force and his close friends	72
11.1	Casimir force between two perfectly conducting plates	74
11.1.1	Case $D = 3$	74
11.1.2	Case $D = 1$ and philosophy about divergent sums	76
11.2	Casimir–Polder force	77
11.3	Orders of forces	79
11.4	The latest advances	80
11.4.1	The dynamical Casimir effect	80
11.4.2	Quantum levitation or repulsive Casimir–Lifshitz forces	80

Notation

- $\mathcal{E}$ is the energy of the system
- $\hat{\mathbf{E}}(\mathbf{r}, t)$, and $\hat{\mathbf{H}}(\mathbf{r}, t)$ are the classical electric and magnetic field vectors
- $\hat{\mathbf{E}}(\mathbf{r}, t)$, and $\hat{\mathbf{H}}(\mathbf{r}, t)$ are the quantum electric and magnetic field *operators*
- $\hat{\mathcal{H}}$ is the quantum hamiltonian of the system
- By ω_0 we normally denote the atomic transition frequency, while the frequency of the field we denote as ω ;
- TLS $\stackrel{\text{def}}{=}$ Two-Level System

Recommended literature

1. Scully, M. O., & Zubairy, M. S. (1999). Quantum optics.
2. Novotny, L., & Hecht, B. (2012). Principles of nano-optics. Cambridge university press.
3. Mandel, L., & Wolf, E. (1995). Optical coherence and quantum optics. Cambridge university press.
4. Fox, M. (2006). Quantum optics: an introduction (Vol. 15). OUP Oxford.
5. Loudon, R., & von Foerster, T. (1974). The quantum theory of light. American Journal of Physics, 42(11), 1041-1042.

1 Atom-field interaction. Semiclassical theory

We start with consideration of a basic problem in our course: the interaction of electromagnetic field with quantum system. In the following we will refer to such system as an "atom". The word *semiclassical* in this context means that we treat electromagnetic field classical, but describe atom as a quantum system. We start with a two-level system (see Fig. 1) as a simplest but very rich example of light-matter interaction.

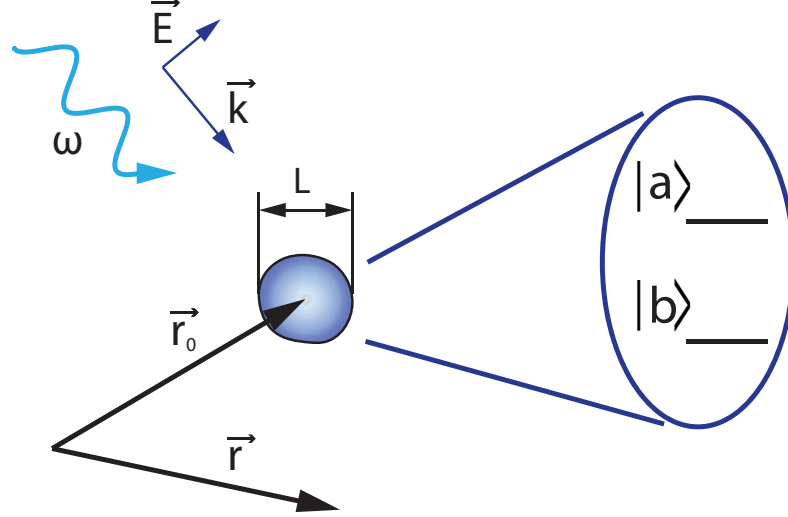


Figure 1: The energy system of two-level atom.

We assume that the electron in the system is initially in the ground state, and at time moment $t = 0$ it is excited with a plane wave with polarization $\mathbf{E}$ and wavevector $\mathbf{k} = \omega/c$. Let us find the probability of atom to be in the excited state at time moment t . In the suggested formulation the problem is a typical example from quantum mechanics course. One should start with writing down the Hamiltonian $\hat{\mathcal{H}}_0$ of an electron in the system without electric field :

$$\hat{\mathcal{H}}_0 = \frac{\hat{\mathbf{p}}^2}{2m} + \hat{V}(\mathbf{r}). \quad (1.1)$$

Here $\hat{\mathbf{p}}$ is the momentum operator, and $\hat{V}$ is the potential energy of the electron.

One can find the eigen states and energy levels of the atom:

$$\hat{\mathcal{H}}_0 |\psi\rangle = \mathcal{E} |\psi\rangle. \quad (1.2)$$

In the following we will assume that there are two eigen states, which we define as $|a\rangle$ and $|b\rangle$ (ground state) with energies E_a and E_b correspondingly. Their difference gives the energy of atomic transition $\hbar\omega_0 = \mathcal{E}_a - \mathcal{E}_b$. The Hamiltonian of the system after introducing the electromagnetic field is as follows:

$$\hat{\mathcal{H}} = \frac{(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A})^2}{2m} + \hat{V}(\mathbf{r}) - e\varphi. \quad (1.3)$$

NB: Potentials are not determined uniquely, and gauge transformation may take place:

$$\mathbf{A} \rightarrow \mathbf{A} + \frac{\hbar c}{e} \nabla \chi, \quad \varphi \rightarrow \varphi - \frac{\hbar c}{e} \frac{\partial \chi}{\partial t}, \quad (1.4)$$

where $\chi(\mathbf{r}, t)$ is a real-valued function of coordinate and time. Then the wave function should also be transformed.

$$\psi \rightarrow \psi e^{i\chi(\mathbf{r}, t)}. \quad (1.5)$$

Next, we write down a vector potential describing the incident plane wave:

$$\mathbf{A} = \mathbf{A}_0 e^{i\mathbf{k}\mathbf{r} - i\omega t} = \mathbf{A}_0(t) e^{i\mathbf{k}\mathbf{r}}. \quad (1.6)$$

Assuming that characteristic scale of the system L is much smaller than the wavelength

$$L \ll \lambda \quad (1.7)$$

one can expanding $\mathbf{A}$ near $\mathbf{r}_0$ (the radius-vector of atom center) in Taylor series:

$$\mathbf{A}(\mathbf{r}, t) \approx \mathbf{A}_0(t) e^{i\mathbf{k}\mathbf{r}_0} (1 + i\mathbf{k}\mathbf{p}) \approx \mathbf{A}_0 e^{i\mathbf{k}\mathbf{r}_0}. \quad (1.8)$$

So the vector potential does not change in space, but in time. By choosing the coordinate so that $\mathbf{r}_0 = 0$ one can simplify the system even further $\mathbf{A} \approx \mathbf{A}_0(t)$:

$$\hat{\mathcal{H}} = \frac{1}{2m} \left(\hat{\mathbf{p}} - \frac{e}{c} \mathbf{A}_0(t) \right)^2 + V(\mathbf{r}) + e\varphi. \quad (1.9)$$

The standard choice it to use Coulomb gauge:

$$\text{div } \mathbf{A} = 0, \quad \varphi = 0. \quad (1.10)$$

After that, using $\hat{\mathbf{p}} = -i\hbar\nabla$, we obtain

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2m} \left(\nabla - \frac{ie}{\hbar c} \mathbf{A}_0 \right)^2 + V(\mathbf{r}). \quad (1.11)$$

Our goal is to find out the temporal evolution of the system. This is can be done by solving the Schrödinger equation:

$$\widehat{\mathcal{H}}\psi = i\hbar \frac{\partial \psi}{\partial t}$$

First of all, we simplify the Hamiltonian by introducing a new wave function $\tilde{\psi} = \underbrace{\psi e^{-i\frac{e}{\hbar c} \mathbf{A}_0 \mathbf{r}}}_{\hookrightarrow u}$. This substitution and related unitary transformation $\widehat{\mathcal{H}} \rightarrow u^\dagger \widehat{\mathcal{H}} u$ will make an momentum shift $\hat{\mathbf{p}} - e/c\mathbf{A} \rightarrow \hat{\mathbf{p}}$. Indeed, if one recalls that

$$(\nabla - i\mathbf{g})(\nabla - i\mathbf{g})\tilde{\psi} e^{i\mathbf{g}\mathbf{r}} = (\nabla - i\mathbf{g})(\nabla \tilde{\psi}) e^{i\mathbf{g}\mathbf{r}} = (\nabla^2 \tilde{\psi}) e^{i\mathbf{g}\mathbf{r}}, \quad (1.12)$$

where $\mathbf{g} \stackrel{\text{def}}{=} \frac{e}{\hbar c} \mathbf{A}_0(t)$, the Schrödinger equation will give us

$$\hat{\mathcal{H}}\psi = i\hbar \frac{\partial \psi}{\partial t} \Rightarrow \underbrace{\left(-\frac{\hbar^2}{2m} \nabla^2 + V \right)}_{\hat{\mathcal{H}}_0} \tilde{\psi}(\mathbf{r}, t) = i\hbar \frac{\partial \tilde{\psi}}{\partial t} - \hbar \mathbf{r} \cdot \frac{\partial \mathbf{g}}{\partial t} \tilde{\psi}. \quad (1.13)$$

Lets pay attention to the second term in right and side

$$\frac{\partial \mathbf{g}}{\partial t} = \frac{e}{\hbar c} \frac{\partial \mathbf{A}_0(t)}{\partial t} = -\frac{e}{\hbar} \mathbf{E}(t). \quad (1.14)$$

Using that we can rewrite (1.13) as

$$\hat{\mathcal{H}}_0 \tilde{\psi} - \underbrace{e\mathbf{r}\mathbf{E}\tilde{\psi}}_{\text{atom-field interaction}} = i\hbar \frac{\partial \tilde{\psi}}{\partial t}. \quad (1.15)$$

The second component in the lefthand side is the interaction term, which appeared after transformation of the Hamiltonian. By its form one can treat it as a dipole energy in the electric field, and we will define $\hat{\mathcal{H}}_1 \stackrel{\text{def}}{=} -\mathbf{d} \cdot \mathbf{E}$, and $\mathbf{d} \stackrel{\text{def}}{=} -e\mathbf{r}$ is the dipole moment. This, finally, leads us to a simplified form of the Schrödinger equation (we omit the tilde sign here to avoid additional idle symbols):

$$\left(\widehat{\mathcal{H}}_0 + \widehat{\mathcal{H}}_1\right) \tilde{\psi} = i\hbar \frac{\partial \tilde{\psi}}{\partial t}. \quad (1.16)$$

In order to solve this equation one can apply the expansion of $|\psi\rangle$ over the eigenstates of non-perturbed system:

$$|\tilde{\psi}\rangle = C_a(t) |a\rangle + C_b(t) |b\rangle, \quad (1.17)$$

where

$$\hat{\mathcal{H}}_0 |a\rangle = \mathcal{E}_a |a\rangle, \quad \hat{\mathcal{H}}_0 |b\rangle = \mathcal{E}_b |b\rangle. \quad (1.18)$$

NB: Here we switch to "bra" and "ket" representation.

We recall that $\mathcal{E}_a - \mathcal{E}_b = \hbar\omega_0$ is the transition frequency. We assume that the incident field has following time dependence $\mathbf{E} = \mathbf{E}_0 \cos \omega t$, so $\hat{\mathcal{H}}_1 = -\mathbf{d} \cdot \mathbf{E}$. Let us assume that initially the system is in the ground state in accordance with the formulation of the problem:

$$|\tilde{\psi}\rangle \Big|_{t=0} = |b\rangle \quad \rightarrow \quad \begin{aligned} C_a(0) &= 0, \\ C_b(0) &= 1. \end{aligned} \quad (1.19)$$

Then one can rewrite the equation in the form:

$$(E_a C_a |a\rangle + E_b C_b |b\rangle) - \mathbf{d} \cdot \mathbf{E} C_a |a\rangle - \mathbf{d} \cdot \mathbf{E} C_b |b\rangle = i\hbar \dot{C}_a |a\rangle + i\hbar \dot{C}_b |b\rangle. \quad (1.20)$$

Projecting it over $|a\rangle$ and $|b\rangle$ one can get:

$$\begin{cases} \langle a| \cdot (1.20) : & E_a C_a - C_a \langle a| \mathbf{d} \cdot \boldsymbol{\varepsilon} |a\rangle - C_b \langle a| \mathbf{d} \cdot \boldsymbol{\varepsilon} |b\rangle = i\hbar \dot{C}_a, \\ \langle b| \cdot (1.20) : & E_b C_b - C_b \langle b| \mathbf{d} \cdot \boldsymbol{\varepsilon} |b\rangle - C_a \langle b| \mathbf{d} \cdot \boldsymbol{\varepsilon} |a\rangle = i\hbar \dot{C}_b. \end{cases} \quad (1.21)$$

In the dipole approximation the field $\mathbf{E}$ does not change in space, so we can write

$$\langle a| \mathbf{d} \cdot \mathbf{E} |a\rangle = \langle a| \mathbf{d} |a\rangle \cdot \mathbf{E}, \quad \langle a| \mathbf{d} \cdot \mathbf{E} |b\rangle = \langle a| \mathbf{d} |b\rangle \cdot \mathbf{E}. \quad (1.22)$$

This allows one to introduce dipole matrix elements:

$$\mathbf{d}_{\alpha\beta} \stackrel{\text{def}}{=} \langle \alpha| \mathbf{d} |\beta\rangle = \int dV \psi_\alpha^*(\mathbf{r}) e\mathbf{r} \psi_\beta(\mathbf{r}) \quad \alpha, \beta = a, b. \quad (1.23)$$

By symmetry considerations it follows $|\mathbf{d}_{\alpha\alpha}| \ll |\mathbf{d}_{\alpha\beta}|_{\alpha \neq \beta}$, since normally neighbouring states have opposite parity, and $e\mathbf{r}$ is the odd function. Basing on this, we set $\mathbf{d}_{aa}, \mathbf{d}_{bb} \rightarrow 0$ in (1.21), and we can write

$$\begin{cases} i\hbar \dot{C}_a = \mathcal{E}_a C_a - C_b \mathbf{d}_{ab} \cdot \mathbf{E}, \\ i\hbar \dot{C}_b = \mathcal{E}_b C_b - C_a \mathbf{d}_{ba} \cdot \mathbf{E}. \end{cases} \quad \mathbf{d}_{ba} = \mathbf{d}_{ab}^* \quad (1.24)$$

Here we can see that if we "turn off" the interaction ($\mathbf{d}_{\alpha\beta} = 0$) states $|a\rangle$ and $|b\rangle$ will be unloosened. If we turn the interaction on transitions will take place.

To get rid off phase factor we introduce

$$\tilde{C}_a(t) \stackrel{\text{def}}{=} C_a(t) e^{-\frac{i}{\hbar} E_a t}, \quad \tilde{C}_b(t) \stackrel{\text{def}}{=} C_b(t) e^{-\frac{i}{\hbar} E_b t}. \quad (1.25)$$

After that we have

$$\begin{cases} \dot{\tilde{C}}_a = \frac{i}{\hbar} \mathbf{d}_{ab} \cdot \mathbf{E} e^{i\omega_0 t} \tilde{C}_b(t), \\ \dot{\tilde{C}}_b = \frac{i}{\hbar} \mathbf{d}_{ab}^* \cdot \mathbf{E} e^{-i\omega_0 t} \tilde{C}_a(t). \end{cases} \quad (1.26)$$

Now we need to apply *rotating wave approximation* (RWA). To understand what is it lets look at

$$\mathbf{E}(t) e^{i\omega_0 t} = \mathbf{E}_0 \frac{e^{i\omega t} + e^{-i\omega t}}{2} e^{i\omega_0 t} = \frac{1}{2} \mathbf{E}_0 \left(\underbrace{e^{i(\omega_0 + \omega)t}}_{\text{fastly oscillating}} + e^{i(\omega_0 - \omega)t} \right). \quad (1.27)$$

Fast oscillating term will lead to a small contribution, so it may be neglected. Leaving only $\sim e^{i(\omega_0 - \omega)t}$ term is called RWA.

It is convenient to denote $\Delta \stackrel{\text{def}}{=} \omega_0 - \omega$, which is the frequency of detuning between the excitation frequency and atomic transition frequency, which gives us:

$$\begin{cases} \dot{\tilde{C}}_a = \frac{i}{2\hbar} \mathbf{d}_{ab} \cdot \mathbf{E}_0 e^{i\Delta t} \tilde{C}_b(t), \\ \dot{\tilde{C}}_b = \frac{i}{2\hbar} \mathbf{d}_{ab}^* \cdot \mathbf{E}_0 e^{-i\Delta t} \tilde{C}_a(t). \end{cases} \quad (1.28)$$

One can see that there is pronounced coupling between the ground and excited state, which is defined by constant

$$\Omega_R \stackrel{\text{def}}{=} \frac{|\mathbf{d}_{ab} \cdot \mathbf{E}_0|}{\hbar}, \quad (1.29)$$

which is called *Rabi frequency*. Its real part gives the strength of coupling between the states. To illustrate our result one can consider zero detuning case $\Delta = 0$ (resonant excitation) and a specific phase of incident light:

$$\begin{cases} \dot{\tilde{C}}_a = i \frac{\Omega_R}{2} \tilde{C}_b, \\ \dot{\tilde{C}}_b = i \frac{\Omega_R}{2} \tilde{C}_a. \end{cases} \quad (1.30)$$

The solution for given initial conditions (1.19) is easy to obtain: $\tilde{C}_b = i \cos\left(\frac{\Omega_R t}{2}\right)$, $\tilde{C}_a = i \sin\left(\frac{\Omega_R t}{2}\right)$. The squared amplitudes of the coefficients $|C_a|^2$ and $|C_b|^2$ have the real physical meaning of the probability of occupation on excited and ground states respectively.

NB: *There are couple of very simple but illustrative conclusion one can make:*

1. *If $\mathbf{d} \perp \mathbf{E}$, then $\Omega_R = 0$ and there is no coupling between the two states;*
2. *To increase Rabi frequency one needs to increase the amplitude of incident wave $|\mathbf{E}_0|$;*
3. *The inversion population oscillates with period $\frac{2\pi}{\Omega_R}$, however the wavefunction, which describes the quantum state is periodic with $\frac{4\pi}{\Omega_R}$;*
4. *There is no spontaneous emission in the system. If one turns out the field at time moment t then the system will remain in its state.*

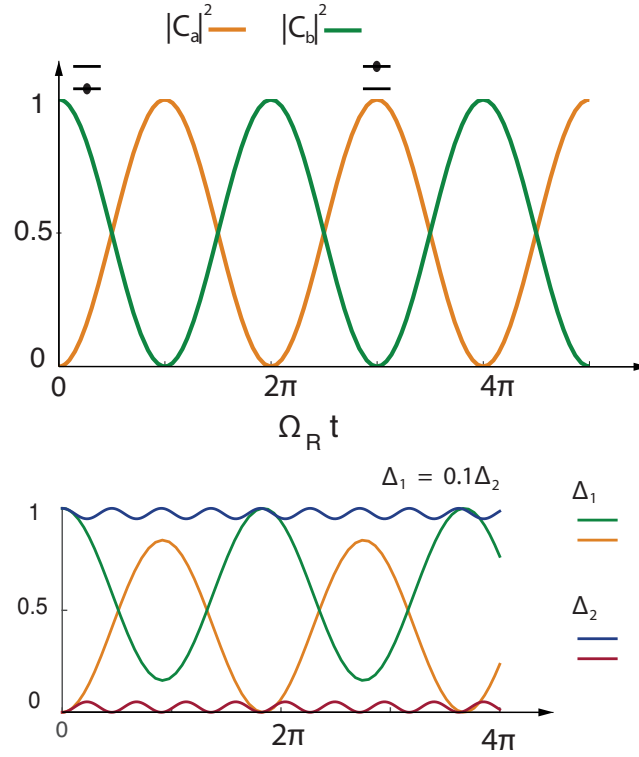


Figure 2: Rabi oscillations of two-level system. Could you add to this solution a dynamics in the non-resonant excitation.

Homework. Deadline: 1st December

1. (3 pts) The hydrogen atom wave functions with quantum numbers (n, l, m_l) of $(2, 0, 0)$ and $(3, 1, 0)$ are as follows:

$$\psi_1(r, \theta, \phi) = \frac{1}{4\sqrt{2\pi}a_0^{3/2}} \left(2 - \frac{r}{a_0} \right) \exp^{-r/2a_0}$$

$$\psi_2(r, \theta, \phi) = \frac{\sqrt{2}}{81\sqrt{\pi}a_0^{5/2}} \left(6 - \frac{r}{a_0} \right) r \cos(\theta) \exp^{-r/3a_0}$$

where a_0 is the Bohr radius, where θ is the angle with the z -axis (quantization axis). Calculate the intensity of z -polarized light field and light power required to observe Rabi oscillations with period of several picoseconds.

2. Consider a TLS from the first lecture. Assume that at time moment $t = 0$ the system was in ground state, and it was excited by a finite optical pulse given by an amplitude envelope:

$$\mathbf{E}_0(t) = \mathbf{E}_{env}(t) \cos(\omega t),$$

where $\mathbf{E}_{env}(t)$ is a slowly varying function. Please, formulate the condition for the envelope function, which ensure that the system will be in the inversed population state after the pulse has passed.

3. (6 pts) Consider a two-level system with non-zero diagonal elements of dipole transition operator:

$$\hat{d} = \begin{pmatrix} d_{aa} & d_{ab} \\ d_{ab}^* & d_{bb} \end{pmatrix}$$

Using semiclassical approach generalize the RWA (Rotating wave approximation) for this case and find the expression for “new” Rabi frequency.

Hint: To simplify the problem use **the interaction picture**:

$$1) \hat{H}_0 \neq \hat{H}_0(t) \quad \hat{H} = \hat{H}_0 + \hat{V} \longrightarrow \hat{V}_{int} = e^{-\frac{i}{\hbar} \hat{H}_0 t} \hat{V} e^{\frac{i}{\hbar} \hat{H}_0 t}$$

$$2) \hat{H}_0 = \hat{H}_0(t) \quad \hat{H} = \hat{H}_0(t) + \hat{V} \longrightarrow \hat{V}_{int} = e^{-\frac{i}{\hbar} \int_0^t \hat{H}_0(\tau) d\tau} \hat{V} e^{\frac{i}{\hbar} \int_0^t \hat{H}_0(\tau) d\tau}$$

Isidor Isaac Rabi (29 July 1898 - 11 January 1988)

I encourage you to add some bio here. On you taste :)

Wiki: "In 1942 Oppenheimer attempted to recruit Rabi and Robert Bacher to work at the Los Alamos Laboratory on a new secret project. They convinced Oppenheimer that his plan for a military laboratory would not work, since a scientific effort would need to be a civilian affair. The plan was modified, and the new laboratory would be a civilian one, run by the University of California under contract from the War Department. In the end, Rabi still did not go west, but did agree to serve as a consultant to the Manhattan Project.[52] Rabi attended the Trinity test in July 1945. The scientists working on Trinity set up a betting pool on the yield of the test, with predictions ranging from total dud to 45 kilotons of TNT equivalent (kt). Rabi arrived late and found the only entry left was for 18 kilotons, which he purchased. Wearing welding goggles, he waited for the result with Ramsey and Enrico Fermi. The blast was rated at 18.6 kilotons, and Rabi won the pool."

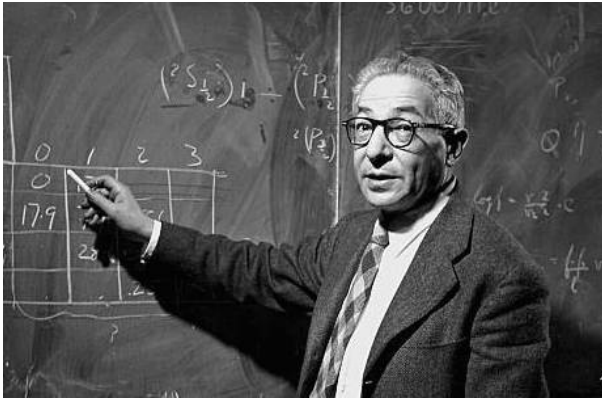


Figure 3: Isidor Isaac Rabi (29 July 1898 - 11 January 1988)

2 Density matrix of two energy level system

We continue the consideration of a TLS started previous lecture. In Section 1 we obtained the dynamics of a TLS, showing Rabi oscillations. There is no need to stress that any dissipative forces were omitted in that consideration. This results in a conclusion that turning off the incident field will make the system to "freeze" in the final state forever (see Fig. 4). This contradicts with the well-known effect of spontaneous relaxation, happening in the real life. One could add a phenomenological dissipative terms to dynamical system Eq. (1.30), but this will ruin the wavefunction and it's normalization, as we change the final equations but not the Hamiltonian of the system. In the framework of semiclassical consideration the only correct way out is to use *density matrix formalism*.

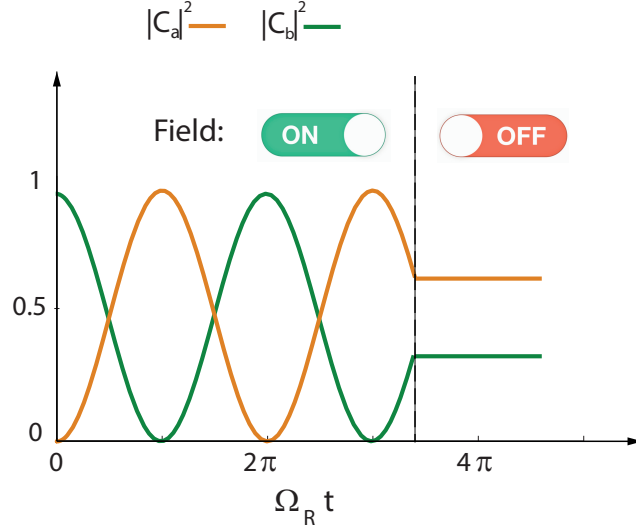


Figure 4: An illustrative case of Fig. 2 when the incident field is turned off.

A density matrix is a matrix that describes a quantum system in a mixed state, a statistical ensemble of several quantum states. In other words, density matrix is very useful if we do not know a wave function which contains all the information of the system but we still want to describe our system.

We illustrate this with two very typical examples (fig. 5):

1. $|\psi_A\rangle$ is known but we need to describe only B -system which is a subsystem of A . This means that though the system is in a pure quantum mechanical state, we might not always can describe the subsystem B with any particular wavefunction.
2. B — is not a conservative system, and there is an interaction with energy exchanging with the outer system (reservoir) is taking place. We do not know the exact mechanism of this interaction, and, thus, can not build a proper wavefunction. But still can give some assumptions on that.

NB: *If a state can be described by a particular wavefunction, then we call it "pure" state*

If $|\psi\rangle$ is known than density matrix is defined by

$$\hat{\rho} = |\psi\rangle \langle\psi| \quad (2.1)$$

If we expand to Fock states $|\psi\rangle = \sum C_n |n\rangle$ then

$$\hat{\rho} = \sum_{nm} C_n^* C_m |m\rangle \langle n|. \quad (2.2)$$

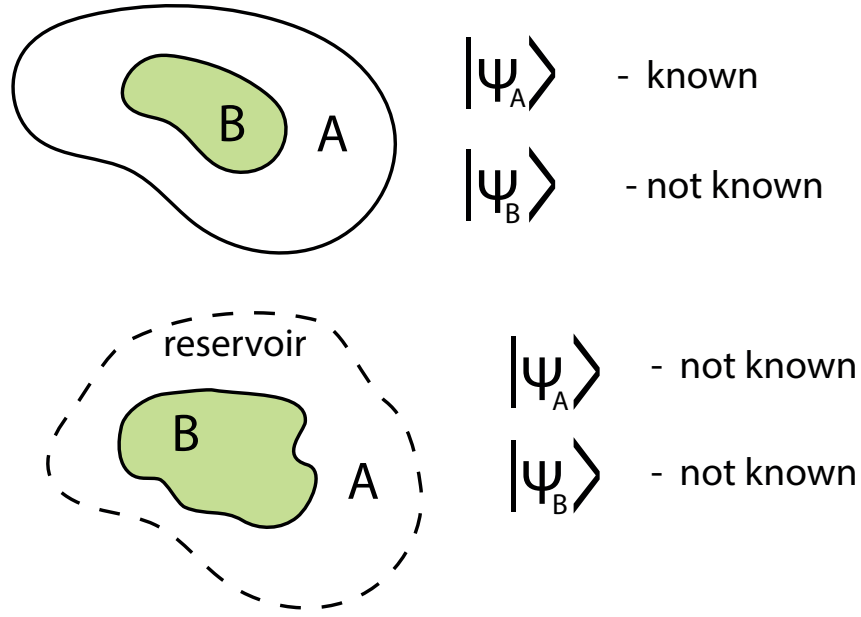


Figure 5: Possible systems where one needs to use density matrix

Matrix elements — $\rho_{mn} = C_n^* C_m$.

Example: Density matrix for two-level system.

For a two-level system considered in the first lecture we have:

$$|\psi\rangle = C_a |a\rangle + C_b |b\rangle, \quad (2.3)$$

so

$$\hat{\rho} = |\psi\rangle \langle \psi| = \begin{pmatrix} |C_a(t)|^2 & C_a(t)C_b^*(t) \\ C_b(t)C_a^*(t) & |C_b(t)|^2 \end{pmatrix}, \quad \rho_{ij} = \langle i | \hat{\rho} | j \rangle. \quad (2.4)$$

Mean operator value:

$$\bar{f} = \langle \psi | \hat{f} | \psi \rangle = \sum_{nm} C_n^* C_m \langle n | \hat{f} | m \rangle = \sum_{nm} \rho_{nm} f_{nm} = \text{Tr} \left(\hat{\rho} \hat{f} \right). \quad (2.5)$$

This is the main intended use of density matrix! Properties:

1. Hermiticity: $\hat{\rho}^\dagger = \hat{\rho}$.
2. $\text{Tr} \hat{\rho} = 1 \rightarrow$ diagonal elements: $\rho_{nn} = |C_n|^2$. Remark: only for pure states!
3. $\hat{\rho} = |\psi\rangle \langle \psi| \rightarrow \hat{\rho}^2 = \hat{\rho}$. It's easy to show:

$$\hat{\rho}^2 = |\psi\rangle \underbrace{\langle \psi | \psi \rangle}_{\rightarrow 1} \langle \psi| = |\psi\rangle \langle \psi|. \quad (2.6)$$

It also means that density matrix of pure state is a *projector*.

2.1 Density matrix of a subsystem

Let $\hat{f}_A$ is an operator that acts only in subsystem A . Now let us ask a question: how to find mean value $\bar{f}_A$?

Subsystem A does not have its own wave function and $|\psi\rangle$ (wave function of $A + B$) does not fall in multiplication of $|\psi_A\rangle$ and $|\psi_B\rangle$. But we can write an expansion in eigenfunctions $|n\rangle$ of subsystem A and $|\alpha\rangle$ of B :

$$|\psi\rangle = \sum_{n\alpha} C_{n\alpha} |n\rangle |\alpha\rangle. \quad (2.7)$$

Now we can write

$$\begin{aligned} \bar{f}_A &= \langle\psi| \hat{f}_A |\psi\rangle = \sum_{n\alpha n'\alpha'} C_{n\alpha} C_{n'\alpha'}^* \overbrace{\langle\alpha'| \langle n'| \hat{f}_A | n\rangle |\alpha\rangle}^{f_{n'n}} = \\ &= \sum_{n\alpha n'\alpha'} C_{n\alpha} C_{n'\alpha'}^* f_{n'n} \underbrace{\langle\alpha'| \alpha\rangle}_{\leftrightarrow=\delta_{ij}\alpha\alpha'} = \sum_{nn'} \sum_{\alpha} C_{n\alpha} C_{n'\alpha}^* f_{n'n} = \sum_{nn'} (\hat{\rho}_A)_{n'n} f_{n'n}. \end{aligned} \quad (2.8)$$

Here we denoted

$$(\hat{\rho}_A)_{n'n} = \sum_{\alpha} C_{n\alpha} C_{n'\alpha}^* \quad (2.9)$$

or in other words

$$(\hat{\rho}_A)_{n'n} = \sum_{\alpha=\alpha'} (\hat{\rho}_{A+B})_{n\alpha n'\alpha'} \cdot \quad (2.10)$$

In the general case:

$$\boxed{\hat{\rho}_A = \text{Tr}_B (\hat{\rho}_{A+B})} \quad (2.11)$$

2.2 Density matrix of a mixed state

Now let us consider a *mixed state*. Let there are lots of systems in different pure states $|\psi_i\rangle$. Let there are N_i particles in $|\psi_i\rangle$, then the whole amount of particles (=systems) is $N = \sum N_i$. The probability to find any system in $|\psi_i\rangle$ is $w_i = \frac{N_i}{N}$. The question we are trying to answer now: how can we find mean operator value?

$$\bar{f} = \text{Tr} \hat{\rho} \hat{f} = ? \quad \text{since} \quad \hat{\rho} = ? \quad (2.12)$$

Scenario is the following:

1. Calculation of quantum-average values: $f_i = \langle\psi_i| \hat{f} |\psi_i\rangle$.
2. Calculation of classical average values: $\bar{f} = \sum w_i f_i$.

So

$$\begin{aligned} \bar{f} &= \sum w_i \langle\psi_i| \hat{f} |\psi_i\rangle = \sum_i w_i \sum_{nm} \underbrace{(C_n^i)^* C_m}_{\rho_{mn}^i} \underbrace{\langle n| \hat{f} | m\rangle}_{f_{nm}} = \\ &= \sum_{nm} \sum_i \rho_{mn}^i w_i f_{nm} \stackrel{\text{def}}{=} \text{Tr} (\hat{\rho} \hat{f}). \end{aligned} \quad (2.13)$$

Here we defined density matrix as $(\hat{\rho}_{\text{mix}})_{mn} = \sum_i w_i \rho_{mn}^i$. Density matrix has a probability meaning:

$$\hat{\rho}_{\text{mix}} = \sum_i w_i \hat{\rho}_i, \quad \hat{\rho}_i = |\psi_i\rangle \langle\psi_i|. \quad (2.14)$$

Coefficients w_i are defined by *statistical (not quantum!) mechanics* (e.g. Boltzmann distribution).

Remarks:

1. $\text{Tr } \hat{\rho}_{\text{mix}} = 1$ by virtue of the fact that $\sum w_i = 1$.
2. $\text{Tr } \hat{\rho}_{\text{mix}}^2 = \sum w_i^2 \leq 1$.

Proof:

$$\begin{aligned}
\text{Tr } \hat{\rho}_{\text{mix}}^2 &= \text{Tr} \sum_{ij} w_i w_j \sum_{nm} C_{m_i}^* C_{n_i} |n_i\rangle \langle m_i| \sum_{n'm'} C_{m'_j}^* C_{n'_j} |n'_j\rangle \langle m'_j| = \\
&= \text{Tr} \sum_{ij} w_i w_j \sum_{nmn'm'} C_{m_i}^* C_{m'_j}^* C_{n_i} C_{n'_j} |n_i\rangle \langle m_i| |n'_j\rangle \langle m'_j| = \\
&\quad \underbrace{\langle m_i | |n'_j\rangle \langle m'_j |}_{\hookrightarrow = \delta_{mn'} \delta_{ij}} = \\
&= \text{Tr} \sum_i w_i^2 \underbrace{\sum_{n'} |C_{n'}|^2}_{\hookrightarrow = 1} \hat{\rho}_i = \sum_i w_i^2. \quad (2.15)
\end{aligned}$$

It means it is not a projector anymore! Testing criterion of pureness is introduced by

$$\boxed{\mu = \text{Tr } \hat{\rho} - \text{Tr } \hat{\rho}^2 \geq 0} \quad (2.16)$$

2.3 Density matrix of a two-level system

Let us consider a TLS with upper state $|a\rangle$ with energy $E_a = \hbar\omega_a$ and lower $|b\rangle$ with $E_b = \hbar\omega_b$ shown in Fig. ???. Wave function of such system is

$$|\psi\rangle = C_a |a\rangle + C_b |b\rangle, \quad |a\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |b\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (2.17)$$

Density matrix is

$$\hat{\rho} = \begin{pmatrix} |C_a(t)|^2 & C_a(t)C_b^*(t) \\ C_b(t)C_a^*(t) & |C_b(t)|^2 \end{pmatrix}. \quad (2.18)$$

The von Neumann equation for time evolution

$$\dot{\hat{\rho}} = -\frac{i}{\hbar} [\hat{\mathcal{H}}, \hat{\rho}]. \quad (2.19)$$

It is convenient to separate Hamiltonian into two parts $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_1$, where

$$\hat{\mathcal{H}}_0 = \hbar\omega_a |a\rangle \langle a| + \hbar\omega_b |b\rangle \langle b|, \quad (2.20)$$

$$\hat{\mathcal{H}}_1 = -E(t) (d_{ab} |a\rangle \langle b| + d_{ab}^* |b\rangle \langle a|), \quad (2.21)$$

where $E(t) = E_0 \cos \omega t$. Motion equations

$$\dot{\rho}_{mn} = -\frac{i}{\hbar} \sum_k (H_{mk} \rho_{kn} - \rho_{mk} H_{kn}) \quad (2.22)$$

or

$$\begin{cases} \dot{\rho}_{aa} = -\frac{i}{\hbar} E(t) (d_{ab}^* \rho_{ab} - d_{ab} \rho_{ba}) \\ \dot{\rho}_{bb} = -\dot{\rho}_{aa} \\ \dot{\rho}_{ab} = -i\omega_0 \rho_{ab} + i \frac{E(t) d_{ab}}{\hbar} (\rho_{bb} - \rho_{aa}) \\ \dot{\rho}_{ba} = (\dot{\rho}_{ab})^* \end{cases} \quad (2.23)$$

where $\omega_0 = \omega_a - \omega_b$. Needless to note this is not a RWA yet. Let $\rho_{ab} = \tilde{\rho}_{ab} e^{-i\omega t}$ and $\rho_{ba} = \tilde{\rho}_{ba} e^{i\omega t}$ then

$$\begin{aligned}\dot{\tilde{\rho}}_{ab} &= -i(\omega_0 - \omega)\tilde{\rho}_{ab} + i\frac{E(t)d_{ab}e^{i\omega t}}{\hbar}(\rho_{bb} - \rho_{aa}) = -i(\omega_0 - \omega)\tilde{\rho}_{ab} + \\ & i\frac{E_0 d_{ab}}{2\hbar}(e^{i\omega t} + e^{-i\omega t})e^{i\omega t}(\rho_{bb} - \rho_{aa}) \approx -i\underbrace{(\omega_0 - \omega)}_{\Delta}\tilde{\rho}_{ab} + i\underbrace{\frac{E_0 d_{ab}}{2\hbar}}_{\Omega_R/2}(\rho_{bb} - \rho_{aa}).\end{aligned}\quad (2.24)$$

So in RWA we have

$$\begin{cases} \dot{\tilde{\rho}}_{ab} = -i\Delta\tilde{\rho}_{ab} + i\frac{\Omega_R}{2}(\rho_{bb} - \rho_{aa}), \\ \dot{\tilde{\rho}}_{ba} = (\tilde{\rho}_{ab})^*, \\ \dot{\rho}_{aa} = -i\frac{\Omega_R}{2}(\tilde{\rho}_{ab} - \tilde{\rho}_{ba}), \\ \dot{\rho}_{bb} = -\dot{\rho}_{aa}. \end{cases}\quad (2.25)$$

While obtaining this we have also assumed that $d_{ab} \in \mathbb{R}$, which simplifies the consideration without losing the generality.

Let us consider a specific case: at time $t = 0$ system is in the lower state:

$$\begin{aligned}C_a &= 0, & C_b &= 1, \\ \rho_{bb}|_{t=0} &= 1, & \rho_{aa}|_{t=0} &= 0.\end{aligned}\quad (2.26)$$

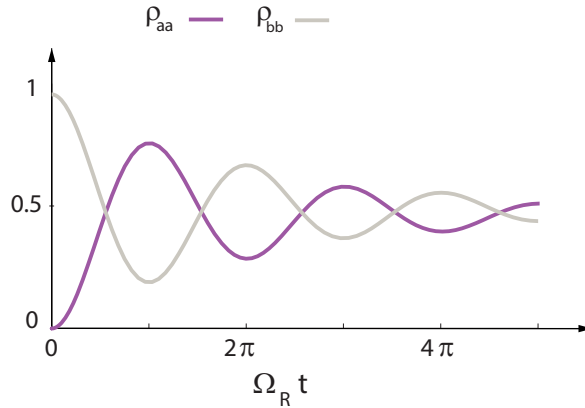


Figure 6: Inverse population of two level system for resonant and non-resonant excitation

Let the incident field be $\mathbf{E} = \mathbf{E}_0 \cos(\omega t + \varphi)$. Induced dipole moment is $\mathbf{d}_{ab} = \langle a | e \mathbf{r} | b \rangle \in \mathbb{C}$, so $|\Omega| = \Omega_R = \frac{\mathbf{E}_0 \cdot \mathbf{d}_{ab}}{\hbar}$.

2.4 Bloch sphere

There is an easy-to-see way to visualize the state of a two-level system. First of all, while we are considering the a system in a pure state we have only *two independent parameters* because of the additional restrictions conditions applied for the density matrix. It appears that it is convenient to introduce the following variables:

$$\begin{cases} x = 2 \operatorname{Re} \rho_{ab} \\ y = 2 \operatorname{Im} \rho_{ab} \\ z = \rho_{aa} - \rho_{bb} \end{cases}\quad (2.27)$$

These coordinates form a new vector $\mathbf{r} = \{x, y, z\}$. There are several properties of this vector:

- if the system is in the pure state, then the $|r| = 1$ and the vector's end lies on the sphere, which is often called *Bloch's sphere*. This is a consequence of the relation $|r|^2 = \text{Tr}(\hat{\rho}^2)$;
- by the definition, z -coordinate shows the inverse population of the system;
- one can show, that the system (2.30) can be rewritten in the form of

$$\dot{\mathbf{r}} = [\boldsymbol{\Omega} \times \mathbf{r}], \quad (2.28)$$

where $\boldsymbol{\Omega}$ is the vector of precession (see Homework).

2.5 Dissipations

We recall that one of the main reasons, which stimulated us to consider the density matrix, was the potential possibility of introducing the losses and dissipation processes in the system. This can be done phenomenologically or more rigorously using relaxation theory (see Scully&Zubairy for the details). The simplified conclusion of this rigorous approach can be formulate in a general form of recipe of adding the losses through the dissipation operator $\hat{\Gamma}$:

$$\dot{\hat{\rho}} = -\frac{i}{\hbar} [\hat{\mathcal{H}}, \hat{\rho}] - \frac{1}{2} \{ \hat{\Gamma}, \hat{\rho} \}, \quad (2.29)$$

where $\Gamma_{mn} = \gamma_n \delta_{mn}$, γ_n is the dissipation rate constants. However, this is not the only way how one can introduce the dissipation terms.

2.5.1 Spontaneous emission of TLS

With the density matrix tool one can introduce the process of spontaneous emission for describing the TLS. This can be done by adding the necessary terms in the right-hand side of the master equation. As soon as we want to describe the spontaneous decay of the excited state, we should add a terms $-\gamma_1 \rho_{aa}$ in the equation describing the dynamics of the upper state population given by ρ_{aa} . In order to conserve the total probability, one should add the same term to the equation for ρ_{bb} . We will also add a rate of coherence dissipation defined by the constant γ_2

$$\begin{cases} \dot{\rho}_{aa} = -i\frac{\Omega_R}{2} (\tilde{\rho}_{ab} - \tilde{\rho}_{ba}) - \gamma_1 \rho_{aa}, \\ \dot{\rho}_{bb} = -\dot{\rho}_{aa}, \\ \dot{\tilde{\rho}}_{ab} = -i\Delta \tilde{\rho}_{ab} + i\frac{\Omega_R}{2} (\rho_{bb} - \rho_{aa}) - \gamma_2 \tilde{\rho}_{ab}, \\ \dot{\tilde{\rho}}_{ba} = (\dot{\tilde{\rho}}_{ab})^*. \end{cases} \quad (2.30)$$

The introduced dissipation rates γ_1 and γ_2 are often referred to inversion population decay and coherence decay (dephasing) rates respectively. The are also often called *longitudinal* and *transverse* damping constants, the reason for this notation will be clear later. The inversion population rate has quite simple origin and can be related to many processes, which can be divided into radiative (spontaneous emission) and non-radiative(relaxation through phonon interaction or any other non-radiating channel): $\gamma_1 = \gamma_1^r + \gamma_1^{nr}$. The de-coherence rate is, actually, more complicated process and it can not be lower that doubled inversion population decay rate:

$$\gamma_2 = 2\gamma_1 + \gamma_2'.$$

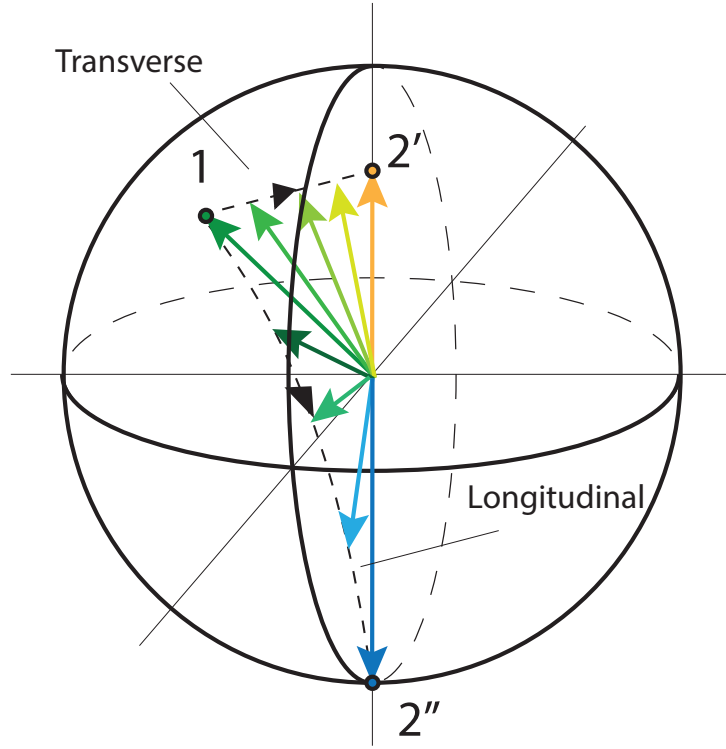


Figure 7: The trajectory of the state on the Bloch sphere for two cases: $\gamma'_2 \gg \gamma_1$ strong transverse damping (1-2'), $\gamma''_2 \ll \gamma_1$ strong longitudinal damping (1-2'').

This illustrates the fact that the inversion population results in decoherence of the quantum state. On the other hand, the factor γ'_2 corresponds to so-called *pure dephasing* time, which is not necessarily related to population decay. Finally, all the relaxation processes result in the destroying of the pure quantum states and transferring it into the mixed state. This can be very easily illustrated with the help of the Bloch sphere. We consider two distinct cases:

1. $\gamma'_2 \gg \gamma_1$ corresponds to pure dephasing process. In this case the population inversion stays constant at this time scale, and the relaxation process occurs with $z \approx \text{const}$ and the vector has only transverse dynamics (see Fig. 7 1-2').
2. $\gamma_1 \ll \gamma'_2$ corresponds to dynamics with varying all the coordinates. The final state corresponds to the ground state as shown in Fig. 7 1-2'' trajectory.

Note, that in both cases the vector leaves the surface and goes inside the sphere, which is a sign of an incoherence, and the system transfers switches into a mixed state.

2.6 Dielectric constant of media

The proposed picture of a TLS interaction with classical field based on density matrix gives a powerful tool for analyzing large quantum systems consisting of many atoms. This allows, in particular, to build a proper description of lasing. But here we will consider another important example of deriving the dielectric susceptibility of a media by considering it as an ensemble of identical TLS. From classical electrodynamics one knows that susceptibility is a constant, which ties together polarization vector $\mathbf{P}$ and electric amplitude vector $\mathbf{E}$:

$$\mathbf{P} = \chi \mathbf{E}.$$

At the same time, the polarization vector is a dipole moment of a unit volume, which can be calculated basing on the quantum mechanical approach. The time-varying polarization can be obtained by averaging the dipole moment of a TLS using the density matrix of the system:

$$\tilde{\mathbf{P}}(t) = N \text{Tr}(\hat{\rho}(t)\hat{\mathbf{d}}) = N(\rho_{ab}d_{ab}^* + \rho_{ba}d_{ab}), \quad (2.31)$$

here N is the concentration of atoms in media. On the other hand temporal dependence of $\tilde{\mathbf{P}}(t)$ can be expanded in two counter-rotating terms:

$$\tilde{\mathbf{P}}(t) = \mathbf{P}e^{-i\omega t} + \mathbf{P}^*e^{i\omega t}, \quad (2.32)$$

which gives us that $\mathbf{P} = N\tilde{\rho}_{ab}d_{ab}^*$. Now, in order to derive the expression for polarization tensor one should consider the stationary regime, which forms in TLS after applying external harmonic field excitation. This can be easily done by assuming $\dot{\hat{\rho}} \rightarrow 0$ in Eq. (2.30), which results in

$$\begin{cases} \rho_{aa} = -i\frac{\Omega_R}{2\gamma_1}(\tilde{\rho}_{ab} - \tilde{\rho}_{ba}) \\ \tilde{\rho}_{ab} = i\frac{\Omega_R}{2(\gamma_2 + i\Delta)}(\rho_{bb} - \rho_{aa}) \end{cases} \quad (2.33)$$

Using the properties of the density matrix, we get $\rho_{bb} - \rho_{aa} = 1 - 2\rho_{aa} = 1 + \frac{i\Omega_R}{\gamma_1}(\tilde{\rho}_{ab} - \tilde{\rho}_{ba})$. After some simple algebra, one obtains:

$$\rho_{bb} - \rho_{aa} = 1 - \frac{\Omega_R^2\gamma_2}{\gamma_1(\gamma_2^2 + \Delta^2)}(\rho_{bb} - \rho_{aa}),$$

which, finally, gives us:

$$\rho_{bb} - \rho_{aa} = \frac{\gamma_2^2 + \Delta^2}{\gamma_2^2 + \Delta^2 + \frac{\Omega_R^2\gamma_2}{\gamma_1}}, \quad (2.34)$$

and

$$\tilde{\rho}_{ab} = \frac{1}{2} \frac{\Omega_R(\Delta + i\gamma_2)}{\gamma_2^2 + \Delta^2 + \frac{\Omega_R^2\gamma_2}{\gamma_1}}. \quad (2.35)$$

Now, recalling the definition of Rabi frequency $\Omega_R = d_{ab}E/\hbar$, we can write down the expression for χ :

$$\chi(\omega) = \frac{|d_{ab}|^2((\omega_0 - \omega) + i\gamma_2)}{2\hbar \left((\omega_0 - \omega)^2 + \gamma_2^2 + \frac{\Omega_R^2\gamma_2}{\gamma_1} \right)}. \quad (2.36)$$

The obtained expression for $\chi(\omega)$ demonstrates two very important features:

- Any quantum dipole transition gives a resonant response in material constants of media.
- Any TLS is highly nonlinear material. Indeed, as $\Omega_R^2 \sim E^2$, then the susceptibility is a function of field intensity. Moreover, at the resonance the imaginary part $\text{Im}(\chi)$, responsible for absorption has following expression:

$$\text{Im}(\chi(\omega)) = \frac{|d_{ab}|^2\gamma_1}{2\hbar(\gamma_1\gamma_2 + \Omega_R^2)}.$$

2.7 Homework

Homework. Можно здесь добавить домашку про построение сферы Блоха с затуханием и, например, указать сразу пример системы с $\Gamma \neq 0$.
бла бла бла

Homework. Density Matrix of TLS

1. (4 pt) Entangled spins

The system consisting of two electrons forming a singlet state have the following wavefunction

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$$

Find the density matrix for the first electron and show that the state of the electron is not pure.

2. (5 pt) Dynamics on the Bloch sphere

A problem of a general two-level system can be expressed in terms of 2×2 matrices, which can be expanded in the following basis: $\{\hat{1}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$. For example, the Hamiltonian and the density matrix can be expressed as

$$\begin{aligned}\hat{H} &= \frac{\hbar}{2} (\omega_0 \hat{1} + \boldsymbol{\Omega} \cdot \boldsymbol{\sigma}) \\ \hat{\rho} &= \frac{1}{2} (r_0 \hat{1} + \mathbf{r} \cdot \boldsymbol{\sigma})\end{aligned}$$

As soon as $\text{Tr}(\hat{\rho}) = 1$, $r_0 = 1$ and we can set ω_0 to 0 by choosing a zero-level energy. Proof the equivalence of the Neumann equation $\dot{\hat{\rho}} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}]$ and $\dot{\mathbf{r}} = \boldsymbol{\Omega} \times \mathbf{r}$. Write several sentences commenting this result.

3. Damped TLS

Consider a problem of a two-level system interacting with electromagnetic field using a semiclassical approach. Now we can phenomenologically introduce decoherence:

$$\begin{aligned}\dot{\rho}_{aa} &= -\frac{i\Omega_R}{2}(\rho_{ba} - \rho_{ab}) - 2\gamma\rho_{aa} \\ \dot{\rho}_{bb} &= -\dot{\rho}_{aa} \\ \dot{\rho}_{ba} = \dot{\rho}_{ab}^* &= \frac{i\Omega_R}{2}(\rho_{bb} - \rho_{aa}) + [i(\omega_0 - \omega) - \gamma]\rho_{ba}\end{aligned}$$

Assume that the system is initially prepared in the ground state ($\rho_{bb}(t=0) = 1$)

- (5 pt) Solve the system of ODE numerically and analyze it for i) strong ($\Omega_R \gg \gamma$) and weak ($\Omega_R \ll \gamma$) coupling, ii) on/off- resonant excitation;
- (2 pt) Find the analytical solution for populations ρ_{aa} and ρ_{bb} for $t \rightarrow \infty$ (stationary regime).
- [Optional task] (6pt) Be means of any mathematical software plot the trajectory of the solution obtained in a) on the Bloch sphere

3 Secondary quantization

The discussed semi-classical approach is a powerful method for description of light-matter interaction. It allows explanation of many complex physical effects. By adding classical fluctuation into the semi-classical system can cover a bigger part of quantum optics [Mandel]. However, there are several important effects, which can not be explained in terms of semi-classical approach. Spontaneous emission of excited atom is one of them, though it lies in the fundamentals of many physical systems. In order to understand this and other purely quantum effects, one needs to introduce a quantum of electromagnetic field, which are *photons*, or, in other words, one should *quantize field*. In order to do that, we will follow the standart method of field quantization and start from wave nature of electromagnetic field.

3.1 Vector potential of the electromagnetic field

The Maxwell equations in vacuum in the absence of charges and currents have very simple form:

$$\left\{ \begin{array}{l} \text{rot } \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, \\ \text{rot } \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \\ \text{div } \mathbf{E} = 0, \\ \text{div } \mathbf{H} = 0. \end{array} \right. \quad \begin{array}{l} (3.1a) \\ (3.1b) \\ (3.1c) \\ (3.1d) \end{array}$$

We will work with vector potential $\mathbf{A}$, which can be introduced as follows:

$$\mathbf{H} = \text{rot } \mathbf{A} \quad (3.2)$$

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi, \quad (3.3)$$

where φ is the scalar electric potential. The vector and scalar potential can be defined in non-unique way up to the gradient of an arbitrary real function and time derivative of the same function, which often called “calibration freedom”. In order to eliminate this uncertainty in $\mathbf{A}$ and φ we apply additional restriction (Lorentz gauge):

$$\text{div } \mathbf{A} = 0. \quad (3.4)$$

Substituting the electric field expression to the (3.1b) gives

$$\text{rot rot } \mathbf{A} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{1}{c} \nabla \frac{\partial \varphi}{\partial t} \quad (3.5)$$

and since

$$\text{rot rot } \mathbf{A} = \underbrace{\text{grad div } \mathbf{A} - \text{div grad } \mathbf{A}}_{\hookrightarrow=0} = -\Delta \mathbf{A} \quad (3.6)$$

we arrive to the Helmholtz equation for the vector potential:

$$\Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{1}{c} \nabla \frac{\partial \varphi}{\partial t}. \quad (3.7)$$

Applying the divergence operation over (3.3), one can get the equation for the scalar potential:

$$\underbrace{\text{div } \mathbf{E}}_{\hookrightarrow=0} = -\frac{1}{c} \frac{\partial}{\partial t} \underbrace{\text{div } \mathbf{A}}_{\hookrightarrow=0} - \Delta \varphi \quad \Rightarrow \quad \Delta \varphi = 0 \quad \Rightarrow \quad \nabla \varphi = 0. \quad (3.8)$$

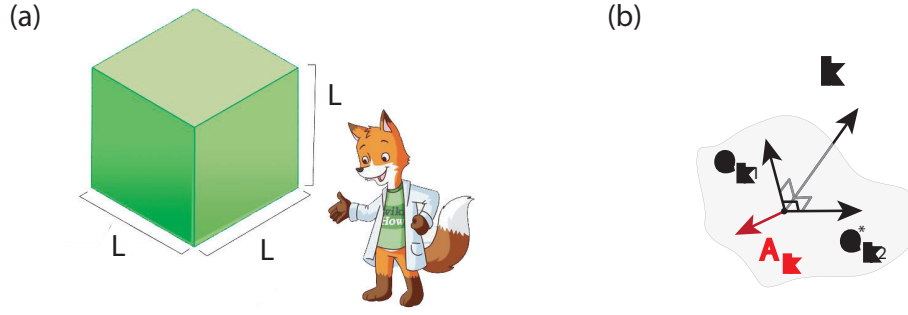


Figure 8: Formulation of the problem

In a free space, we can apply the scalar potential $\varphi \equiv 0$, which simplifies the system for $\mathbf{A}$:

$$\begin{cases} \Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \\ \text{div } \mathbf{A} = 0. \end{cases} \quad (3.9)$$

3.2 Field in the box, harmonics expansion, and the energy of the electromagnetic field

After we set the equation for vector potential, we can build its solution for a very simple yet very important case. Let us consider a cube box with the length of the edge L (Fig. 8) with periodic boundary conditions. Such a box can interpret a free space once the limit $L \rightarrow \infty$ will be applied.

The solution of (3.9) may be written as a sum of all eigen solutions, which are the plane waves in Cartesian system. Taking into the account periodic boundary that results in

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}}(t) e^{i\mathbf{k}\mathbf{r}}, \quad k_{\alpha} = \frac{2\pi n_{\alpha}}{L}, \quad \alpha = x, y, z \quad n_{\alpha} \in \mathbb{Z}. \quad (3.10)$$

- The vector potential $\mathbf{A}(\mathbf{r}, t)$ is real valued that provides the condition:

$$\mathbf{A}_{\mathbf{k}}(t) = \mathbf{A}_{-\mathbf{k}}^*(t) \quad (3.11)$$

- The temporal dependence of the vector potential is described by two oscillating terms:

$$\mathbf{A}_{\mathbf{k}}(t) = \mathbf{c}_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} + \mathbf{c}_{-\mathbf{k}}^* e^{i\omega_{\mathbf{k}} t}, \quad (3.12)$$

where $\omega_{\mathbf{k}} = ck = c\sqrt{k_x^2 + k_y^2 + k_z^2}$. The form of $\mathbf{A}_{\mathbf{k}}(t)$ is provided by the condition (3.11).

- The Lorentz gauge leads to the fact that the *waves are transverse*:

$$\text{div } \mathbf{A} = 0 \quad \rightarrow \quad \sum_{\mathbf{k}} \mathbf{k} \mathbf{A}_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} = 0 \quad \Longleftrightarrow \quad \boxed{\mathbf{A}_{\mathbf{k}}(t) \cdot \mathbf{k} = 0}. \quad (3.13)$$

Consider a wave with wave vector $\mathbf{k}$. According to Maxwell equations, there are two independent polarizations, so we introduce two transverse polarization vectors (see Fig. 8

(b)) $\mathbf{e}_{\mathbf{k}1}; \mathbf{e}_{\mathbf{k}2}$. Three vectors $(\mathbf{e}_{\mathbf{k}1}; \mathbf{e}_{\mathbf{k}2}; \mathbf{k}/k)$ form a right-handed orthonormal basis which implies:

$$\mathbf{k} \cdot \mathbf{e}_{\mathbf{k}s} = 0, \quad [\mathbf{e}_{\mathbf{k}1} \times \mathbf{e}_{\mathbf{k}2}^*] = \mathbf{k}/k, \quad (3.14)$$

$$\mathbf{e}_{\mathbf{k}s} \cdot \mathbf{e}_{\mathbf{k}s'}^* = \delta_{ss'}, \quad \mathbf{c}_{\mathbf{k}} = \sum_s c_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s}.$$

After that we can rewrite decomposition of $\mathbf{A}$ as

$$\begin{aligned} \mathbf{A} &= \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} (c_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s} e^{-i\omega_{\mathbf{k}}t} + c_{-\mathbf{k}s}^* \mathbf{e}_{-\mathbf{k}s}^* e^{i\omega_{\mathbf{k}}t}) \cdot e^{i\mathbf{k}\mathbf{r}} = \\ &= / \text{inverse 2nd sum using (3.12): } (-k) \rightarrow k/ = \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} (u_{\mathbf{k}s}(t) \mathbf{e}_{\mathbf{k}s} e^{i\mathbf{k}\mathbf{r}} + u_{\mathbf{k}s}^*(t) \mathbf{e}_{\mathbf{k}s}^* e^{-i\mathbf{k}\mathbf{r}}), \end{aligned} \quad (3.15)$$

where $u_{\mathbf{k}s}(t) = c_{\mathbf{k}s} e^{-i\omega_{\mathbf{k}}t}$. Now we can write fields

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = \frac{i}{c} \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} \omega_{\mathbf{k}} (u_{\mathbf{k}s}(t) \mathbf{e}_{\mathbf{k}s} e^{i\mathbf{k}\mathbf{r}} - u_{\mathbf{k}s}^*(t) \mathbf{e}_{\mathbf{k}s}^* e^{-i\mathbf{k}\mathbf{r}}), \quad (3.16)$$

$$\mathbf{H} = \text{rot } \mathbf{A} = i \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} (u_{\mathbf{k}s} [\mathbf{k} \times \mathbf{e}_{\mathbf{k}s}] e^{i\mathbf{k}\mathbf{r}} - u_{\mathbf{k}s}^* [\mathbf{k} \times \mathbf{e}_{\mathbf{k}s}^*] e^{-i\mathbf{k}\mathbf{r}}). \quad (3.17)$$

The obtained plane-wave expansion allows us to get a simple picture of EM field as an ensemble of oscillators. It is very illustrative to consider the energy of EM field inside the box:

$$\mathcal{H} = \frac{1}{8\pi} \int (\mathbf{H}^2 + \mathbf{E}^2) dV. \quad (3.18)$$

We will simplify this further by using several important relations. First is the orthogonality of the modes:

$$\int_{L^3} e^{i(\mathbf{k}-\mathbf{k}')\mathbf{r}} dV = L^3 \delta_{\mathbf{k}\mathbf{k}'}. \quad (3.19)$$

This feature vanish the $\sum_{\mathbf{k}}$. Second, it is convenient to notice that

$$\mathbf{e}_{\mathbf{k}s}^* \cdot \mathbf{e}_{\mathbf{k}s'} = \delta_{ss'} \quad \rightarrow \quad [\mathbf{k} \times \mathbf{e}_{\mathbf{k}s}^*] \cdot [\mathbf{k} \times \mathbf{e}_{\mathbf{k}s'}] = k^2 \delta_{ss'}. \quad (3.20)$$

Then, we get

$$\mathcal{H} = \frac{L^3}{8\pi} 2 \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}}^2 \left(\underbrace{\frac{\omega_{\mathbf{k}}^2}{c^2} |u_{\mathbf{k}s}|^2}_{\hookrightarrow E^2} + \underbrace{k^2 |u_{\mathbf{k}s}|^2}_{\hookrightarrow H^2} \right), \quad k^2 = \frac{\omega_{\mathbf{k}}^2}{c^2} \quad (\text{for each mode!}) \quad (3.21)$$

$$\mathcal{H} = \frac{L^3}{2\pi} \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}}^2 k^2 |u_{\mathbf{k}s}|^2. \quad (3.22)$$

With this we see that electric and magnetic counterparts give equal contribution into the total EM energy. We split the real and imaginary parts of the mode amplitude $|u_{\mathbf{k}s}|$ by introducing new variables

$$q_{\mathbf{k}s}(t) = u_{\mathbf{k}s}(t) + u_{\mathbf{k}s}^*(t), \quad (3.23)$$

$$p_{\mathbf{k}s}(t) = -i\omega_{\mathbf{k}} (u_{\mathbf{k}s}(t) - u_{\mathbf{k}s}^*(t)). \quad (3.24)$$

It's obvious that

$$u_{\mathbf{k}s}(t) = \frac{1}{2}q_{\mathbf{k}s}(t) - \frac{1}{2i\omega}p_{\mathbf{k}s}(t) \quad \rightarrow \quad |u_{\mathbf{k}s}|^2 = \frac{1}{4\omega_{\mathbf{k}}^2} (p_{\mathbf{k}s}^2 + \omega_{\mathbf{k}}^2 q_{\mathbf{k}s}^2). \quad (3.25)$$

The energy will be as follows

$$\mathcal{H} = \frac{L^3}{4\pi c^2} \sum_{\mathbf{k},s} \frac{\tilde{A}_{\mathbf{k}}^2}{2} (p_{\mathbf{k}s}^2 + \omega_{\mathbf{k}}^2 q_{\mathbf{k}s}^2). \quad (3.26)$$

Let us boldly put $\tilde{A}_{\mathbf{k}} = \sqrt{4\pi c^2/L^3}$, then finally

$$\mathcal{H} = \sum_{\mathbf{k},s} \left(\frac{p_{\mathbf{k}s}^2}{2} + \frac{\omega_{\mathbf{k}}^2 q_{\mathbf{k}s}^2}{2} \right). \quad (3.27)$$

This picture is indeed very illustrative as represent the Hamiltonian of the field as a sum of harmonic oscillators energies.

3.3 Field quantization

The field quantization procedure bases on the quantum-to-classical correspondence principle: we obtain the classical expression, and then quantize it by assuming that physical fields become operators. In our case, we see that the system can be considered as a set of harmonic oscillators, and $\mathcal{H} \sim p^2/2 + \omega^2 q^2/2$, one can correspond quantum operators of coordinate and momentum to classical ones:

$$\begin{cases} q_{\mathbf{k}s} \rightarrow \hat{q}_{\mathbf{k}s}, \\ p_{\mathbf{k}s} \rightarrow \hat{p}_{\mathbf{k}s}. \end{cases} \quad (3.28)$$

We know, that in any quantum oscillator the coordinate and momentum can not commute. Thus, the introduced operators should also satisfy the same commutation relations:

$$[\hat{q}_{\mathbf{k}s}; \hat{p}_{\mathbf{k}'s'}] = i\hbar \delta_{\mathbf{k}\mathbf{k}'}^{(3)} \delta_{ss'}, \quad (3.29)$$

$$[\hat{q}_{\mathbf{k}s}; \hat{q}_{\mathbf{k}'s'}] = [\hat{p}_{\mathbf{k}s}; \hat{p}_{\mathbf{k}'s'}] = 0. \quad (3.30)$$

These operators should also correspond to measurable quantity and, thus, should be hermitian:

$$\hat{q}_{\mathbf{k}s} = \hat{q}_{\mathbf{k}s}^\dagger, \quad \hat{p}_{\mathbf{k}s} = \hat{p}_{\mathbf{k}s}^\dagger. \quad (3.31)$$

Relations (3.29), (3.30) and (3.31) impose conditions for $\hat{q}_{\mathbf{k}s}$ and $\hat{p}_{\mathbf{k}s}$. Then, our Hamiltonian acquires operator form

$$\mathcal{H} \rightarrow \widehat{\mathcal{H}} = \sum_{\mathbf{k},s} \left(\frac{\hat{p}_{\mathbf{k}s}^2}{2} + \frac{\omega_{\mathbf{k}}^2 \hat{q}_{\mathbf{k}s}^2}{2} \right).$$

In order to further simplify the Hamiltonian and consideration in general, it is convenient to introduce the *ladder operators*:

$$\hat{a}_{\mathbf{k}s}(t) = \frac{1}{\sqrt{2\hbar\omega}} (\omega \hat{q}_{\mathbf{k}s} + i\hat{p}_{\mathbf{k}s}), \quad (3.32)$$

and his conjugated friend

$$\hat{a}_{\mathbf{k}s}^\dagger(t) = \frac{1}{\sqrt{2\hbar\omega}} (\omega \hat{q}_{\mathbf{k}s} - i\hat{p}_{\mathbf{k}s}). \quad (3.33)$$

This leads to the useful representation of $\hat{q}_{\mathbf{k}s}$ and $\hat{p}_{\mathbf{k}s}$:

$$\hat{q}_{\mathbf{k}s}(t) = \sqrt{\frac{\hbar}{2\omega}} \left(\hat{a}_{\mathbf{k}s}^\dagger + \hat{a}_{\mathbf{k}s} \right), \quad (3.34)$$

$$\hat{p}_{\mathbf{k}s}(t) = i\sqrt{\frac{\hbar\omega}{2}} \left(\hat{a}_{\mathbf{k}s}^\dagger - \hat{a}_{\mathbf{k}s} \right). \quad (3.35)$$

Commutation relations can be easily derived from consideration $[\hat{q}_{\mathbf{k}s}; \hat{p}_{\mathbf{k}s}]$ in the representation of ladder operators and using (3.29) and (3.30). So we get

$$[\hat{a}_{\mathbf{k},s}; \hat{a}_{\mathbf{k}',s'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}^{(3)} \delta_{ss'}. \quad (3.36)$$

Easy to show that Hamiltonian can be written as follows

$$\hat{\mathcal{H}} = \sum_{\mathbf{k},s} \hbar\omega_{\mathbf{k}} \left[\hat{a}_{\mathbf{k},s}^\dagger \hat{a}_{\mathbf{k},s} + \frac{1}{2} \right]. \quad (3.37)$$

It's convenient to rewrite coefficients $\hat{u}_{\mathbf{k}s}$ and $\hat{u}_{\mathbf{k}s}^+$ as

$$\hat{u}_{\mathbf{k}s} = \frac{1}{2} \left(\hat{q}_{\mathbf{k}s} - \frac{1}{i\omega_{\mathbf{k}}} \hat{p}_{\mathbf{k}s} \right) = \sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}}} \hat{a}_{\mathbf{k}s}, \quad (3.38)$$

$$\hat{u}_{\mathbf{k}s}^+ = \sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}}} \hat{a}_{\mathbf{k}s}^+. \quad (3.39)$$

Now, we can quantize the electric potential and introduce the operator $\mathbf{A} \rightarrow \hat{\mathbf{A}}$ such that:

$$\hat{\mathbf{A}} = \sum_{\mathbf{k},s} A_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s} e^{i\mathbf{k}\mathbf{r}} + \text{h.c.} \right), \quad A_{\mathbf{k}} = \sqrt{\frac{2\pi\hbar c^2}{L^3\omega}}. \quad (3.40)$$

Finally we can write field operators:

$$\hat{\mathbf{E}} = \sum_{\mathbf{k},s} \varepsilon_{\mathbf{k}} \left(i\hat{a}_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s} e^{i\mathbf{k}\mathbf{r}} + \text{e.c.} \right), \quad \varepsilon_{\mathbf{k}} = \sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{L^3}}, \quad (3.41)$$

$$\hat{\mathbf{H}} = \sum_{\mathbf{k},s} A_{\mathbf{k}} \left(i\hat{a}_{\mathbf{k}s} [\mathbf{k} \times \mathbf{e}_{\mathbf{k}s}] e^{i\mathbf{k}\mathbf{r}} + \text{e.c.} \right). \quad (3.42)$$

Remark: Electric field $\mathbf{E}$ satisfies wave equation $\square\mathbf{E} = 0$ at the same time. We could quantize it:

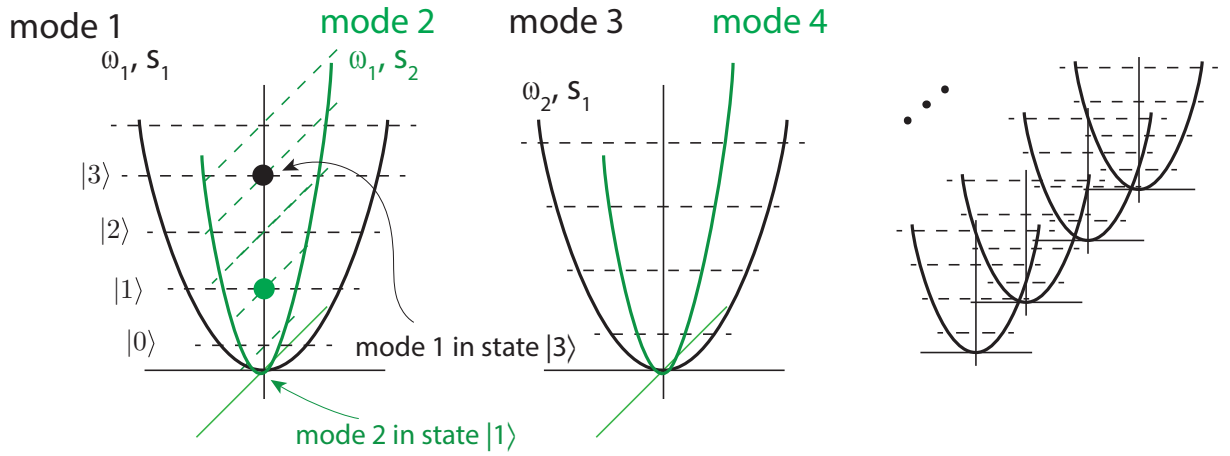
$$\mathbf{E} = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \left[\hat{a}_{\mathbf{k}} \mathbf{e}_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} + \text{c.c.} \right], \quad (3.43)$$

$$\mathbf{H} = \sum_{\mathbf{k}} A_{\mathbf{k}} \left[-i\hat{a}_{\mathbf{k}} [\mathbf{k} \times \mathbf{e}_{\mathbf{k}}] e^{i\mathbf{k}\mathbf{r}} + \text{c.c.} \right]. \quad (3.44)$$

3.4 Ladder operators. Fock state. Second quantization

First of all lets write how ladder operators work:

$$\begin{aligned} \hat{a}_{\mathbf{k}s} |n_{\mathbf{k}s}\rangle &= \sqrt{n_{\mathbf{k}s}} |n_{\mathbf{k}s} - 1\rangle, & \hat{a}_{\mathbf{k}s} |0\rangle &= |0\rangle, \\ \hat{a}_{\mathbf{k}s}^\dagger |n_{\mathbf{k}s}\rangle &= \sqrt{n_{\mathbf{k}s} + 1} |n_{\mathbf{k}s} + 1\rangle, & \hat{a}_{\mathbf{k}s}^\dagger |0\rangle &= |1\rangle. \end{aligned} \quad (3.45)$$



Another important operator is of quantity of particles $\hat{n}_{\mathbf{k}s} = \hat{a}_{\mathbf{k}s}^\dagger \hat{a}_{\mathbf{k}s}$ in $\mathbf{k}s$ mode:

$$\hat{n}_{\mathbf{k}s} |n_{\mathbf{k}s}\rangle = n_{\mathbf{k}s} |n_{\mathbf{k}s}\rangle. \quad (3.46)$$

Another import remark: $|n_{\mathbf{k}s}\rangle$ — it is a state with wave vector $\mathbf{k}$ and state s which has *exactly* n photos.

Let us make things more clear, by discussing a bit about denotations:

$$|n_{\mathbf{k}s}\rangle \longleftrightarrow \psi_n^{\mathbf{k}s}(x). \quad (3.47)$$

Besides, any wave function can decomposed on the basis (if the basis is full):

$$\psi(x) = \sum C_n \varphi_n(x), \quad (3.48)$$

so it's equivalent

$$\psi(x) \longleftrightarrow \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}. \quad (3.49)$$

Column of numbers $(C_1, \dots, C_n)^T$ is a vector in Hilbert space — bra- or ket-vector. For example

$$\left| \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0 \right\rangle \longleftrightarrow \frac{1}{\sqrt{2}} \varphi_1(x) + \frac{1}{\sqrt{2}} \varphi_3(x). \quad (3.50)$$

In particular, Fock state can be written as

$$|n\rangle = \left| 0, \dots, 0, \underbrace{1}_{n\text{-th place}}, 0, \dots, 0 \right\rangle. \quad (3.51)$$

3.5 Fields' fluctuation

Let consider single mode field:

$$\hat{E}_x(0) = \varepsilon \hat{a} + \varepsilon^* \hat{a}^\dagger. \quad (3.52)$$

Let's compute the average field:

$$\langle n | \hat{E}_x | n \rangle = \varepsilon \langle n | \hat{a} | n \rangle + \text{e.c.} = 0, \quad (3.53)$$

because $\langle n | \hat{a} | n \rangle = \sqrt{n} \langle n | n-1 \rangle = 0$.

The average of $\hat{E}^2$ gives us the following

$$\langle n | \hat{E}_x^2 | n \rangle = |\varepsilon|^2 \langle n | \hat{a} \hat{a}^\dagger | n \rangle + |\varepsilon|^2 \langle n | \hat{a}^\dagger \hat{a} | n \rangle = 2 |\varepsilon|^2 \langle n | \hat{n} + \frac{1}{2} | n \rangle = 2 |\varepsilon|^2 \left(n + \frac{1}{2} \right). \quad (3.54)$$

Here we can make a few conclusions: **a)** the average field is zero for any Fock state; **b)** in vacuum state ($n = 0$) *fluctuations are minimal, but not zero!*

A standard calculation procedure of fluctuation of any variable X in $|n\rangle$ state is the following:

$$\Delta X = \sqrt{\langle n | \hat{X}^2 | n \rangle - \langle n | \hat{X} | n \rangle^2}. \quad (3.55)$$

Remark: for many modes we get

$$\begin{aligned} \sum_{\mathbf{k},s} \langle n_{\mathbf{k}s} | \hat{\mathbf{E}}^2 | n_{\mathbf{k}s} \rangle &= \sum_{\mathbf{k},s} 2 |\varepsilon_{\mathbf{k}s}|^2 \left(n_{\mathbf{k}s} + \frac{1}{2} \right) \Big|_{n_{\mathbf{k}s}=0} = \sum_{\mathbf{k},s} 2 |\varepsilon_{\mathbf{k}s}|^2 = \sum_{\mathbf{k},s} \frac{2\pi\hbar\omega_{\mathbf{k}}}{L^3} = \\ &= \sum_{\mathbf{k},s} \frac{\hbar\omega_{\mathbf{k}}}{(2\pi)^2} \Delta k_x \Delta k_y \Delta k_z \Big|_{L \rightarrow \infty} = \frac{2}{(2\pi)^3} \int d^3p \hbar\omega_{\mathbf{k}} = \frac{2 \cdot 4\pi}{(2\pi)^2} \frac{\hbar}{c^3} \int_0^\infty \omega^3 d\omega \rightarrow \infty. \end{aligned} \quad (3.56)$$

So, fluctuations are infinite for multimode vacuum state.

3.6 Homework

1. (2 pt) Calculate the average: $\langle n | (a + a^\dagger)^2 | n \rangle$
2. (2 pt) Calculate the average: $\langle n | (a + a^\dagger)^3 | n - 1 \rangle$
3. (4 pts) **Commutation of EM operators.**

Show the following commutation relation (see Scully&Zubairy):

$$[\hat{E}_j(\mathbf{r}, t), \hat{H}_j(\mathbf{r}', t)] = 0, \quad i = x, y, z$$

$$[\hat{E}_j(\mathbf{r}, t), \hat{H}_k(\mathbf{r}', t)] = -i\hbar c^2 \frac{\partial}{\partial t} \delta^{(3)}(\mathbf{r} - \mathbf{r}'), \quad j, k, l = x, y, z$$

4. (4 pts) **Fock states in the position space.**

Show the operator identities $\hat{a}^+ |n\rangle = \sqrt{n+1} |n+1\rangle$ and $\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$ in the position space.

5. (4 pts) **Quantum virial theorem.**

Consider quantum oscillator in Fock state $|n\rangle$.

- a) Check the virial theorem, computing the average potential $\langle \hat{\Pi} \rangle$ and kinetic $\langle \hat{K} \rangle$ energy
- b) Compute the energy fluctuations $\Delta \Pi$ and ΔK in this state

4 Coherent states

We have already noticed that the Fock states forming a full orthogonal set of function in Hilbert space sometimes lack of physical meaning. In particular, they provide a zero average value of electric field operator $\hat{\mathbf{E}}$. In this lecture we build an alternative set of wave functions, which will acquire the deep physical meaning by the end of the lecture and are called *coherent states*.

4.1 Eigenstates of annihilation operator

The problem with measuring the value of the electric field at the Fock states starts with the problem that the ladder operators in the basis of Fock states have no diagonal elements. The matrix of the $\hat{a}$ operator has the form:

$$\langle m | \hat{a} | n \rangle = \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \sqrt{2} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \cdots \\ 0 & \vdots & \cdots & 0 & \sqrt{n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (4.1)$$

This problem can be overcome if we will switch to the basis of eigen functions of the annihilation operator:

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle.$$

where α is the eigenvalue. Our task is to construct these states $|\alpha\rangle$, so we start with expanding them over the Fock states basis:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle.$$

By substituting the expansion into the eigen equation, one gets

$$\hat{a} |\alpha\rangle = \sum_{n=0}^{\infty} c_n \hat{a} |n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n} |n-1\rangle = \sum_{n=0}^{\infty} \sqrt{n+1} c_{n+1} |n\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle,$$

which gives the recurrent relation

$$c_{n+1} \sqrt{n+1} = c_n \alpha \quad \rightarrow \quad c_n = \frac{\alpha^n}{\sqrt{n!}} c_0 \quad \rightarrow \quad |\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

The constant c_0 can be found from the normalization condition:

$$\langle \alpha | \alpha \rangle = 1 \quad \rightarrow \quad 1 = |c_0|^2 \sum_{n,m=0}^{\infty} \frac{(\alpha^*)^m \alpha^n}{\sqrt{m!n!}} \underbrace{\langle m | n \rangle}_{\hookrightarrow \delta_{mn}} = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^2}{n!}, \quad (4.2)$$

$$c_0 = e^{-|\alpha|^2/2} e^{i\varphi} \quad (4.3)$$

and finally

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{i\varphi} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (4.4)$$

NB: We put the phase $\varphi = 0$ as a wave function can be defined only up to a arbitrary phase factor.

The obtained states are called the *coherent states*. Before going further let's discuss some of their main properties.

4.2 Basic properties of coherent states

- How many photons are there in coherent state?

The average number of photons in the coherent state $|\alpha\rangle$ can be calculated as follows:

$$\bar{n} = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle.$$

From the definition of coherent states

$$\begin{aligned} \hat{a} |\alpha\rangle &= \alpha |\alpha\rangle, \\ \langle \alpha | \hat{a}^\dagger &= \alpha^* \langle \alpha |, \end{aligned}$$

it immediately follows

$$\bar{n} = |\alpha|^2. \quad (4.5)$$

- But how the photons are distributed?

The probability to find n photons in the state $|\alpha\rangle$ state is given by the probability:

$$p_n = |\langle n | \alpha \rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}. \quad (4.6)$$

In expression Eq. 4.6 one can recognize the Poisson distribution with average number of photons $|\alpha|^2$. According to the properties of Poisson distribution the dispersion $\Delta n = |\alpha| \sim \sqrt{\bar{n}}$ and relative fluctuations of photon number is proportional to $\Delta n / \bar{n} \sim \sqrt{\bar{n}}$ average number of photons (Fig. 9). The distribution of photons in a coherent state is shown in Fig.9 for different $\bar{n}$ average number of photons in a coherent state.

- Are the coherent states orthogonal or not?

The answer can be given by direct computation of the scalar product:

$$\begin{aligned} \langle \alpha' | \alpha \rangle &= e^{-|\alpha|^2/2} e^{-|\alpha'|^2/2} \sum \frac{(\alpha'^*)^n \alpha^n}{n!} = e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\alpha'|^2} e^{\alpha'^* \alpha} \neq 0, \Rightarrow \\ |\langle \alpha' | \alpha \rangle|^2 &= e^{-|\alpha - \alpha'|^2}. \end{aligned} \quad (4.7)$$

So the coherent states are not orthogonal, but the amplitude norm of the scalar product exponentially depends on the "distance" between the eigen numbers according to (4.7). For instance, it is very illustrative to consider the following example.

Example:

Consider two states with average number of photon equal to $\bar{n} = 5$ (see Fig.9 (b)). Though the average number of photons in each of them is equal, their scalar product can be quite small:

$$\begin{aligned} |\alpha\rangle &= |3 + 4i\rangle \\ |\alpha'\rangle &= |4 + 3i\rangle \end{aligned} \quad \rightarrow \quad |\langle \alpha' | \alpha \rangle|^2 = e^{-2} \approx 0.1$$

$$\begin{aligned} |\alpha\rangle &= |3 + 4i\rangle \\ |\alpha'\rangle &= |-4 - 3i\rangle \end{aligned} \quad \rightarrow \quad |\langle \alpha' | \alpha \rangle|^2 = e^{-98}.$$

So, effectively, in the latter case one can assume that they are orthogonal.

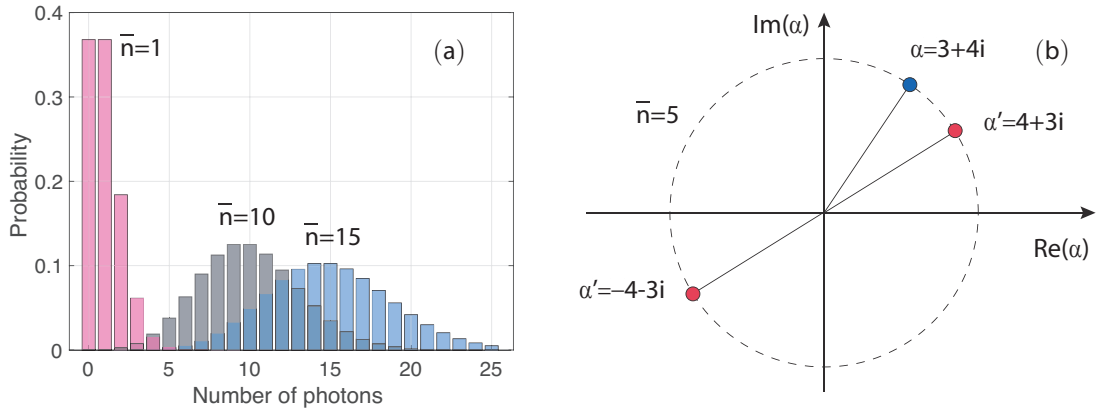


Figure 9: (a) The Probability distribution of photons in the coherent states for different average number of photons $\bar{n}$. (b) The complex plane of eigen values and almost orthogonal states.

Homework. Deadline: 6th of November

1. (3 pt) Construct the eigenfunction of the creation operator $\hat{a}^+|\beta\rangle = \beta|\beta\rangle$.
2. (4 pt) Find the eigenfunctions of a coherent state $|\alpha\rangle$ in the **position** space (either analytically or numerically).

NB: For solving these problems you will require the Baker-Campbell-Hausdorff relations:

- $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}$, if $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{B}, \hat{A}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$
- $e^{i\hat{A}}\hat{B}e^{-i\hat{A}} = \hat{B} + [i\hat{A}, \hat{B}] + \frac{[i\hat{A}, [i\hat{A}, \hat{B}]]}{2!} + \dots$

4.3 Classical field

Let us come back to the problem of the average electric field computation and recall that being computed at the Fock states it gives zero value. The situation becomes different for the coherent states. Indeed, consider a *single mode* field given by the operator:

$$\hat{\mathbf{E}} = \varepsilon (\hat{a}\mathbf{e}e^{i\mathbf{kr}-\omega t} + \text{h. c.}) . \quad (4.8)$$

The average value at the coherent state will be as follows:

$$\mathbf{E} = \langle \alpha | \hat{\mathbf{E}} | \alpha \rangle = \varepsilon \alpha \mathbf{e} e^{i\mathbf{kr}-\omega t} + \text{c. c.} \stackrel{\text{def}}{=} \mathbf{E}_+(\mathbf{r}, t) + \mathbf{E}_-(\mathbf{r}, t). \quad (4.9)$$

The resulting value is a sum of two vector fields corresponding to the plane waves propagating in opposite direction. This is a *classical coherent monochromatic field*, which we are used to. The amplitude and phase of the field is defined by the complex value α :

$$\mathbf{E}_+ = E_+ \mathbf{e} e^{i\mathbf{kr}-\omega t}, \quad E_+ = \alpha \varepsilon, \quad \varepsilon = \sqrt{\frac{2\pi\hbar}{V\omega}}. \quad (4.10)$$

The intensity is proportional to

$$I_+ \propto |E_+|^2 = \varepsilon^2 |\alpha|^2 = \varepsilon^2 \bar{n}, \quad (4.11)$$

while the complex field amplitude also contains the phase φ

$$E_+ = \varepsilon |\alpha| e^{i\varphi}. \quad (4.12)$$

Thus, the coherent state realizes the quantum state, which behaves in a very similar manner to classical field. But this can be understood already at the level of the coherent state definition. Since we have

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle, \quad (4.13)$$

then number of photons in the system does not change under the action of the annihilation operator. In the next lecture we will see that the $\hat{a}$ operator is responsible for detecting (measuring) photons. In classical physics the action of the measuring field does not change the state of the field, and this is exactly what we have for coherent states contrary to Fock states, where annihilation operator changes the number of photons $|n\rangle \rightarrow |n+1\rangle$

Homework. Deadline: 6th of November

1. (3 pt) Prove the overcompleteness of coherent states:

$$\mathbf{1} = \int_{\mathbb{C}} |\alpha\rangle \langle \alpha| \frac{d^2\alpha}{\pi}$$

NB: For solving these problems you will require the Baker-Campbell-Hausdorff relations:

- $e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}$, if $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{B}, \hat{A}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$
- $e^{i\hat{A}} \hat{B} e^{-i\hat{A}} = \hat{B} + [i\hat{A}, \hat{B}] + \frac{[i\hat{A}, [i\hat{A}, \hat{B}]]}{2!} + \dots$

4.4 Fluctuations

The “classical” behaviour of the coherent state is also expressed in the level of the fluctuations, which as low as any quantum state may have. In this paragraph we will elaborate more on that comparing the fluctuations of the coherent states with the Fock states. We start with a recalling the generalized momentum and coordinate operators:

$$\hat{p} = i\sqrt{\frac{\hbar\omega}{2}} (\hat{a}^\dagger - \hat{a}), \quad \hat{q} = \sqrt{\frac{\hbar}{2\omega}} (\hat{a}^\dagger + \hat{a}). \quad (4.14)$$

The general Heisenberg relation tells us that $\Delta p \Delta q \geq \frac{\hbar}{2}$, but now we apply it for Fock and coherent states.

Fock states

To find the fluctuations amplitude let us compute

$$\bar{p} = \langle n | \hat{p} | n \rangle = 0, \quad \bar{q} = \langle n | \hat{q} | n \rangle = 0,$$

and

$$\begin{aligned} \overline{p^2} &= \langle n | \hat{p}^2 | n \rangle = \frac{\hbar\omega}{2} (2n+1), \\ \overline{q^2} &= \langle n | \hat{q}^2 | n \rangle = \frac{\hbar}{2\omega} (2n+1). \end{aligned}$$

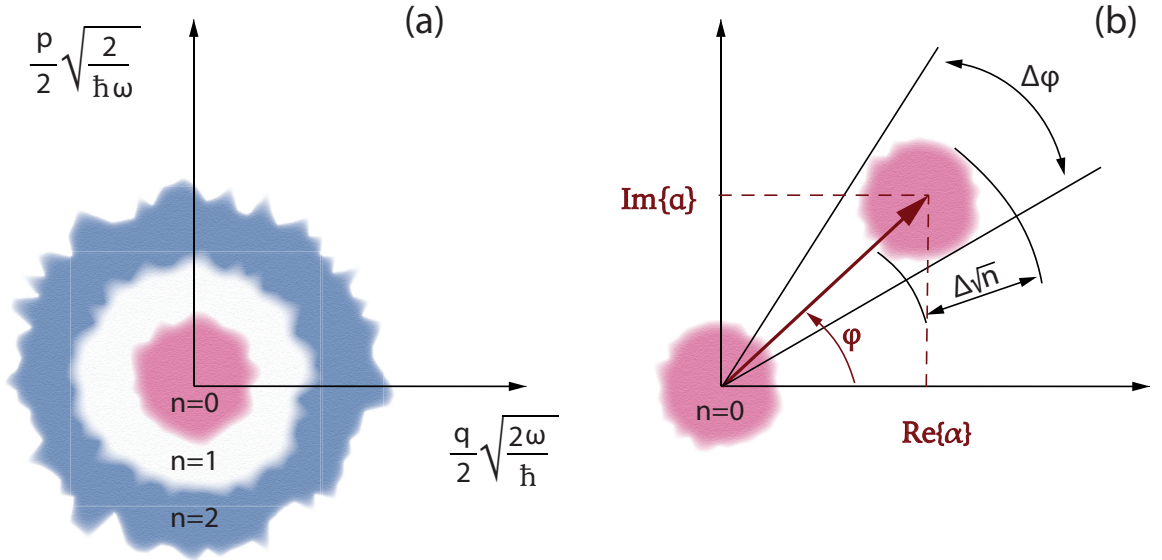


Figure 10: (a) The phase space representation of the Fock state. (b) The phase space representation of the coherent states as a shifted vacuum state

So we have

$$\Delta p_n = \sqrt{p^2 - \bar{p}^2} = \sqrt{\frac{\hbar\omega}{2}} \sqrt{(2n+1)},$$

$$\Delta q_n = \sqrt{q^2 - \bar{q}^2} = \sqrt{\frac{\hbar}{2\omega}} \sqrt{(2n+1)}.$$

Now we can rewrite the uncertainty principle as following

$$\Delta p_n \Delta q_n = \frac{\hbar}{2} (2n+1). \quad (4.15)$$

We need to stress here, that the amplitude of the fluctuation increase with increasing the number of photons in the state. In the phase space the Fock state can be represented by a circle with the centre at the origin and radius increasing with the number of photons in the state (as shown in Fig. 10 (a)).

Coherent states

In a similar manner, we can obtain the fluctuations of the coherent states. For the reasons we have already discussed the average value of the momentum and coordinate operator are non-zero and correspond to imaginary and real part of the eigen value α :

$$\bar{p} = \langle \alpha | \hat{p} | \alpha \rangle = i \sqrt{\frac{\hbar\omega}{2}} (\alpha^* - \alpha) = 2 \sqrt{\frac{\hbar\omega}{2}} \text{Im} \{ \alpha \},$$

$$\bar{q} = 2 \sqrt{\frac{\hbar}{2\omega}} \text{Re} \{ \alpha \},$$

The dispersion can be obtained from the following expression:

$$\overline{p^2} = -\frac{\hbar\omega}{2} \langle \alpha | \hat{a}^{\dagger 2} - \hat{a}^{\dagger} \hat{a} - \hat{a} \hat{a}^{\dagger} + \hat{a}^2 | \alpha \rangle = \frac{\hbar\omega}{2} (4 \text{Im} \{ \alpha \} + 1), \quad (4.16)$$

$$\overline{q^2} = \frac{\hbar}{2\omega} (4 \operatorname{Re} \{\alpha\} + 1). \quad (4.17)$$

Then

$$\Delta p_\alpha = \sqrt{\frac{\hbar\omega}{2} [4 \operatorname{Im} \{\alpha\} + 1 - 4 \operatorname{Im} \{\alpha\}]^{1/2}} = \frac{\hbar\omega}{2}, \quad \Delta q_\alpha = \frac{\hbar}{2\omega}. \quad (4.18)$$

The uncertainty relation for the coherent states will have a simple form of

$$\boxed{\Delta p_\alpha \Delta q_\alpha = \frac{\hbar}{2} \quad \not\sim \quad \alpha.} \quad (4.19)$$

One can see that the uncertainty $\Delta p_\alpha \Delta q_\alpha$ does not depend on α , and it is equal to the vacuum state uncertainty, which is minimal among all the possible Fock states. So the coherent states minimize the uncertainty of the quantum fluctuations.

The phase space representation of the coherent state is shown in Fig. 10 (b) along with the vacuum state. From this illustrative picture one can expect that the coherent state can be generated from a vacuum state with a linear operator $\hat{D}(\alpha)$ which makes some sort of a state shifting in the phase space:

$$|\alpha\rangle = \hat{D}(\alpha) |0\rangle, \quad (4.20)$$

Indeed, the operator $\hat{D}$ exists and is called *the displacement operator*. In order to construct it, one can make trivial operations:

$$e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \hat{D}(\alpha) |0\rangle, \quad (4.21)$$

and since $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle$, so

$$D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}}. \quad (4.22)$$

Remark: factor $e^{-\alpha^* \hat{a}}$ does not change the result (because $e^{-\alpha^* \hat{a}} |0\rangle = |0\rangle$), but gives some additional properties to the $\hat{D}(\alpha)$:

1. Using the Hausdorff relation we can get a compact form of the displacement operator:

$$\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}. \quad (4.23)$$

2. The displacement operator is a unitary operator. It means that $\hat{D}(\alpha) \hat{D}^\dagger(\alpha) = \hat{D}^\dagger(\alpha) \hat{D}(\alpha) = \hat{1}$.
3. Since $\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha)$, the hermitian conjugate of the displacement operator can also be interpreted as a displacement in the opposite “direction”.

4. Following relations hold:

$$\hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) = \hat{a} + \alpha, \quad (4.24)$$

$$\hat{D}(\alpha) \hat{a} \hat{D}^\dagger(\alpha) = \hat{a} - \alpha, \quad (4.25)$$

$$\hat{D}(\alpha) \hat{D}(\beta) = e^{(\alpha \beta^* - \alpha^* \beta)/2} \hat{D}(\alpha + \beta). \quad (4.26)$$

4.5 Squeezed states or getting the maximum accuracy!

The discussed fluctuations behaviour of the Fock and coherent states do not have only abstract fundamental value, but there are very practical consequence. The quantum fluctuations limit the accuracy of the optical measurements: even if you manage to suppress all the sources of systematic error in optical measurement the quantum noise will still be present. Thus, one of the natural question, which may appear: *is it possible to overcome the limit of $\Delta p_\alpha \Delta q_\alpha = \frac{\hbar}{2}$ to get more accurate measurements?*

Sure, we cannot break the uncertainty principle but we can title the balance of scales for ours good. The main idea is to *squeeze* light state in the geometrical meaning making an oval from the circle in the phase space(see Fig. 11). This will give us suppression of the uncertainty at least in one of the directions.

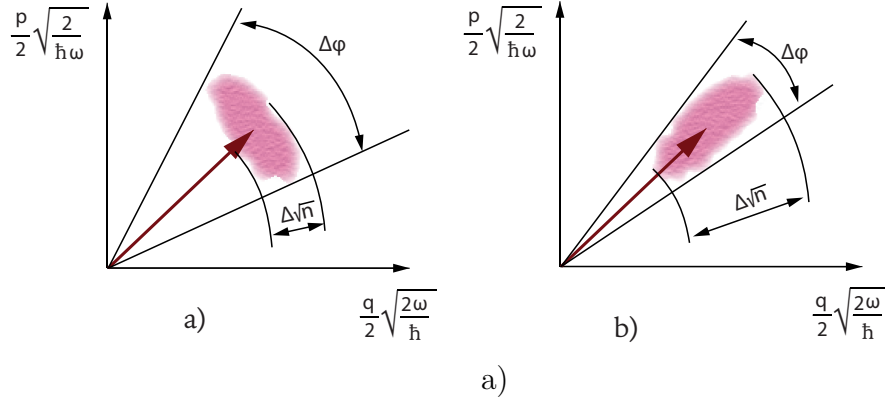


Figure 11: Different light states on a phase plane: (a) amplitude-squeezed light, (b) phase-squeezed light.

To get quantitative and more specific description one can use *the squeeze operator*:

$$\hat{S}(\xi) |\alpha\rangle = |\alpha, \xi\rangle, \quad (4.27)$$

where $|\xi|$ — ratio of the main semiaxes of squeezed state, $\arg \xi$ — a turning angle.

Here are some helpful properties of the squeeze operator:

1. It can be written as following:

$$\hat{S} = e^{\xi \hat{a}^{\dagger 2} - \xi^* \hat{a}^2}. \quad (4.28)$$

2. It is commutative with displacement operator:

$$\hat{S}(\xi) \hat{D}(\alpha) \neq \hat{D}(\alpha) \hat{S}(\xi). \quad (4.29)$$

Another a more explicit view on that can be given with help of a simple consideration. The average value of a single mode field is as follows:

$$E = \varepsilon |\alpha| \cos[\omega t + \varphi]. \quad (4.30)$$

Now, assume that we have fluctuations artificially added to this expression, resulting in:

$$E(t) = \varepsilon (|\alpha| + \delta\alpha) \cos[\omega t + \varphi + \delta\varphi], \quad (4.31)$$

where $\delta\alpha$ and $\delta\varphi$ are the fluctuating components of the amplitude and phase. Within this picture the Fock and coherent states can be visualized in according to Fig. 12 (a).

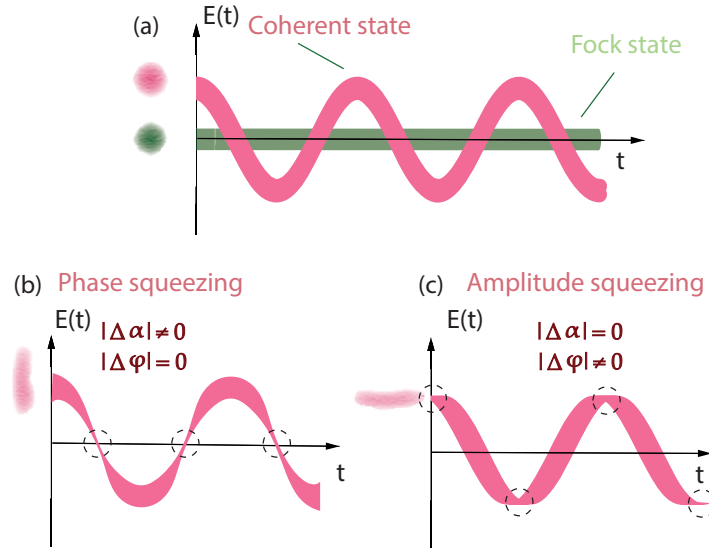


Figure 12: The temporal picture of fluctuating signal: (a) the Fock and coherent states; (b) the phase squeezed state; (c) the amplitude squeezed state.

One can see that coherent state has constant level of fluctuations. Now, if one suppresses the fluctuations of either amplitude or phase the signal will be modified according to Fig. 12 (b) and (c). Measuring them at exact points, one can break the quantum noise limits and measure the signal well below the threshold provided by the Heisenberg principal.

Lets draw attention to (4.28). We can notice that squeeze operator $\hat{S}$ consist of squares of ladder operators ($\sim \hat{a}^{\dagger 2}, \hat{a}^2$). It means here we have a generation of second harmonic ($2\hbar\omega$ instead of $\hbar\omega$). In other words, from a experimenter's point of view, to get the squeezed state one needs a non-linear optical element with $\chi_2 \neq 0$, so it's polarizability $P = \chi_1 E + \chi_2 E^2$.

Homework. Deadline: 6th of November

1. (3 pt) **Eigen states of $\hat{a}^\dagger$**

Construct the eigenfunction of the creation operator $\hat{a}^+|\beta\rangle = \beta|\beta\rangle$.

2. (4 pt) **Coherent states in position space.**

Find the eigenfunctions of a coherent state $|\alpha\rangle$ in the **position** space (either analytically or numerically).

3. (3 pt) **Completeness of coherent states.**

Prove the completeness of coherent states:

$$1 = \int_{\mathbb{C}} |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi}$$

4. (2 pts) **Displacement operator**

Show that the operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ displaces the creation operator by proving the following relations:

$$\hat{D}(\alpha)\hat{a}\hat{D}(\alpha)^{-1} = \hat{a} - \alpha$$

5. (6 pts) **Displacement operator: matrix elements**

Compute the matrix elements $\langle m | \hat{D}(\alpha) | n \rangle$ of the displacement operator in Fock's basis. Express the result in terms of associated Laguerre polynomials.

6. (4 pt) **Classical squeezing.**

For the problem of a classical harmonic oscillator a general solution can be expressed as $x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$, where c_1 and c_2 depend on the initial conditions. Now consider that you drive this system on a $2\omega_0$ frequency so that $V(t) = \frac{1}{2} m \omega_0^2 x^2 (1 + \epsilon \sin 2\omega_0 t)$ ($\epsilon \ll 1$). Prove that $c_1(t) = e^{\beta t}$ and $c_2(t) = e^{-\beta t}$ and find β using the second Newton's law, the method of variation of parameters, considering $c_1(t)$ and $c_2(t)$ as slowly varying variables and ignoring fast-oscillating terms.

7. (5 pt) **Quantum squeezing.**

Now you know that for squeezing you need to drive your system at $2\omega_0$ frequency. In quantum case it corresponds to the process of parametric down-conversion when you have a strong coherent field (mode b) with a frequency $2\omega_0$ as an input and on the output you have two photons of frequency ω_0 (mode a). It can be expressed with the following Hamiltonian $\hat{H} = \hat{a}^\dagger \hat{a}^\dagger \hat{b} + \hat{a} \hat{a} \hat{b}^\dagger$. The corresponding squeezing operator is $\hat{S}(r) = \exp \left[-\frac{r}{2} (\hat{a}^2 - \hat{a}^{\dagger 2}) \right]$, here r describes the strength of squeezing. Compute

$$1) \hat{S}(r) \hat{x} \hat{S}^\dagger(r)$$

$$2) \hat{S}(r) \hat{p} \hat{S}^\dagger(r)$$

where $\hat{x} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}$ and $\hat{p} = \frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2}}$.

NB: For solving these problems you will require the Baker-Campbell-Hausdorff relations:

- $e^{\hat{A} + \hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2} [\hat{A}, \hat{B}]}$, if $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{B}, \hat{A}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$
- $e^{i\hat{A}} \hat{B} e^{-i\hat{A}} = \hat{B} + [i\hat{A}, \hat{B}] + \frac{[i\hat{A}, [i\hat{A}, \hat{B}]]}{2!} + \dots$

5 The coherence of light

The coherence of electromagnetic field is one of the most important characteristic of the light and photon state. It also gives the information about the state of the quantum sources, which emitted the light. Almost all the methods of testing the coherence of light are based on interferometry effects. Thus, we start to consider the coherence of light on the classical examples of light interference.

5.1 Michelson stellar interferometer

One of the important ones is Michelson stellar interferometer invented by the Albert Abraham Michelson in 19th century (see Fig. 13). The basic idea of this interferometer was studying the double stars and the angular distance between them just basing on their optical or microwave signals.

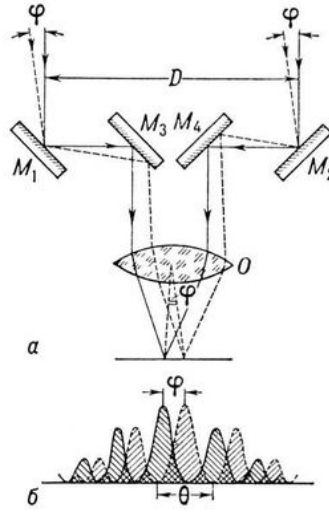


Figure 13: A Michelson stellar interferometer

Lets consider Michelson stellar interferometer (fig 13). From distant stars the two light beams is coming to our set-up. Light is an electromagnetic wave, so we can write:

$$\mathbf{E} = \mathbf{E}_{\mathbf{k}} (e^{i\mathbf{k}\mathbf{r}_{M_1}} + e^{i\mathbf{k}\mathbf{r}_{M_2}}) + \mathbf{E}_{\mathbf{k}'} (e^{i\mathbf{k}'\mathbf{r}_{M_1}} + e^{i\mathbf{k}'\mathbf{r}_{M_2}}). \quad (5.1)$$

If we decide to compute the intensity, we will get (see Scully)

$$I = \langle \mathbf{E} \cdot \mathbf{E}^* \rangle_t = 4I_0 \left(1 + \underbrace{\cos \left[\frac{(\mathbf{k} + \mathbf{k}') \mathbf{r}_0}{2} \right]}_{\text{fast term}} \underbrace{\cos \left[\frac{1}{2} k r_0 \varphi \right]}_{\text{slow term}} \right), \quad (5.2)$$

where $\mathbf{r}_0 = \mathbf{r}_{M_1} - \mathbf{r}_{M_2}$, $I_0 = \langle \mathbf{E}_{\mathbf{k}} \cdot \mathbf{E}_{\mathbf{k}}^* \rangle_t = \langle \mathbf{E}_{\mathbf{k}'} \cdot \mathbf{E}_{\mathbf{k}'}^* \rangle_t$. To make out light and dark spots we need the following condition

$$k r_0 \varphi \approx \pi \quad \rightarrow \quad \varphi = \frac{\pi}{k r_0}. \quad (5.3)$$

It means that to change light spot to the dark one we need $\Delta r_0 = \frac{\pi}{k \varphi}$. So the bigger distance between mirrors M_1 and M_2 , the better resolution we get (we will be able to detect smaller φ). But there appear to be a lot of technical troubles.

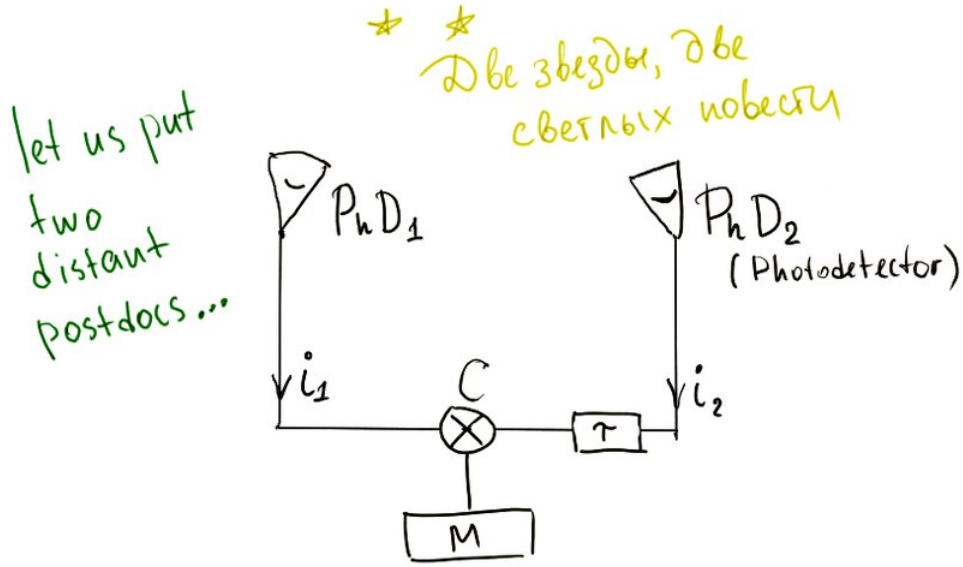


Figure 14: A symmetric interferometer with distant photodetectors

One can choose another pill and build a set up with two distant photodetectors (fig 14). Now we don't have annoying mirrors and measure only currents i_1 and i_2 , which are proportional to the intensities I_1 and I_2 of incident light beams. By this framework we can significantly increase r_0 and this will lead to a higher resolution. People who did very precise measurements, noticed that they no longer measure intensities of incident light, but only noises of almost single photons.

Now let's take a look at this situation from a theoretical point of view. Let us compute the following correlator

$$g^{(2)} = \frac{\langle i_1, i_2 \rangle}{\langle i_1 \rangle \langle i_2 \rangle}, \quad (5.4)$$

where

$$i_1 = \langle i_1 \rangle + \Delta i_1, \quad i_2 = \langle i_2 \rangle + \Delta i_2, \quad \langle i_1 \rangle = \langle i_2 \rangle = i_0. \quad (5.5)$$

Then we can write

$$g^{(2)} = \frac{i_0^2 + \langle \Delta i_1, \Delta i_2 \rangle}{i_0^2} = 1 + \frac{\langle \Delta i_1, \Delta i_2 \rangle}{i_0^2}. \quad (5.6)$$

So in fact one measures the average of Δi_1 and Δi_2 . To study the issue in more details the so called *HBT experiment* was done. The principle scheme of the set up is shown on fig 15. The heart of the matter is that method allows to avoid the atmospherically sensitive terms (fast terms) like $\cos \left[\frac{(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}_0}{2} \right]$.

Let's analyze the second order correlator:

$$g^{(2)}(\tau) = 1 + \frac{\langle \Delta i(t) \Delta i(t + \tau) \rangle}{i_0^2}. \quad (5.7)$$

Here can single out some properties:

1. $g^{(2)}(0) = 1 + \frac{\langle \Delta i^2(t) \rangle}{i_0^2} \geq 1.$
2. $\lim_{\tau \rightarrow \infty} g^{(2)}(\tau) = 1.$
3. $g^{(2)}(0) \geq g^{(2)}(\tau).$

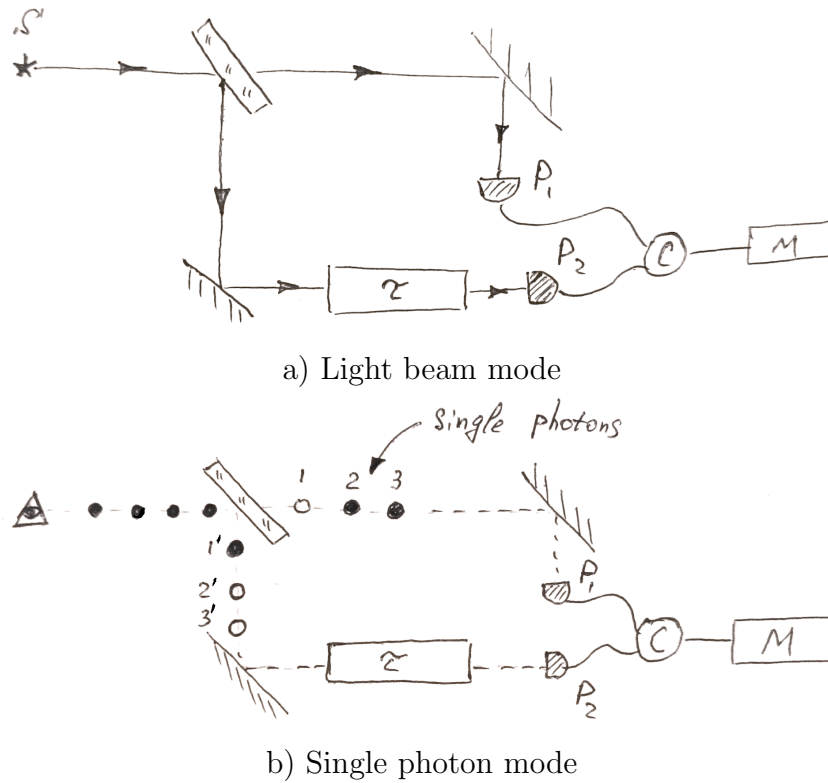


Figure 15: Schematic diagram of the Hanbury Brown-Twiss intensity interferometer. Here P_1 and P_2 are photodetectors, τ is the delay time, C is a multiplier, and M is the integrator

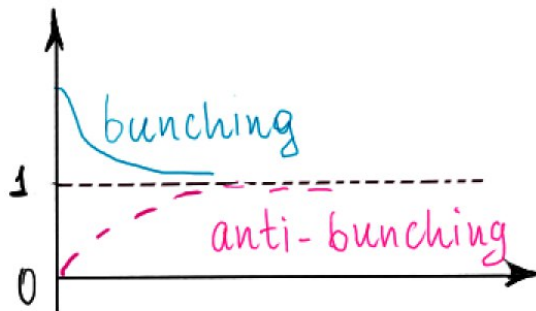


Figure 16: Approximate plot of $g^{(2)}(\tau)$

To summarize we could guess that approximate plot of $g^{(2)}(\tau)$ is as it shown of fig. 16 with a solid line.

During measuring noises, which were mentioned thereinbefore, experimentalists got the next weird result:

$$g^{(2)}(\tau) < 1. \quad (5.8)$$

This effect is called *photon bunching*. But a few lines above it was shown that $g^{(2)}(\tau) \geq 1$. How could it be possible?

Let us consider the same experiments but with a single photons emitter (fig. 15 b). Assume $\tau = 0$, so then we calculate how photon at place 1 is correlated with a photon at place 1'. If our light source emits discrete photos then single photon cannot be at two places at the same time, so $g^{(2)}(0) = 0$.

5.2 Quantum theory of photodetection

Photons are described by field operators

$$\hat{\mathbf{E}} = \hat{\mathbf{E}}^+ + \hat{\mathbf{E}}^-, \quad (5.9)$$

where

$$\hat{\mathbf{E}}^+ = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \hat{a}_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r} - i\omega t} \mathbf{e}_{\mathbf{k}}, \quad (5.10)$$

$$\hat{\mathbf{E}}^- = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\mathbf{r} + i\omega t} \mathbf{e}_{\mathbf{k}}^*. \quad (5.11)$$

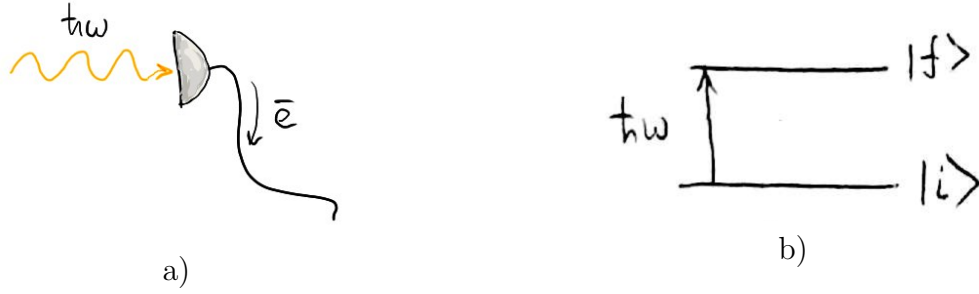


Figure 17: a) An incident photon generate an electron which brings current; b) Atom states

To simplify all the calculations let us consider only *one mode* and only *one linear polarisation* of field:

$$\hat{\mathbf{E}}^+ = \hat{E}^+ \mathbf{e}_{\mathbf{k}}. \quad (5.12)$$

When an incident photon got absorbed, it excites the atom detector: $|i\rangle \rightarrow |f\rangle$ (see fig. 17, b).

The probability of absorption of one photon (= the probability of detection of one photon) depends only on operator $\hat{E}^+$ due to detection principle (see fig. 17). The transition probability of the detector atom for absorbing a photon from the field at position $\mathbf{r}$ between times t and $t + dt$ is proportional to $\omega_1(\mathbf{r}, t)dt$, with

$$\omega_1(\mathbf{r}, t) = \sum_f \left| \langle f | \hat{E}^+ | i \rangle \right|^2 = \sum_f \langle i | \hat{E}^- | f \rangle \langle f | \hat{E}^+ | i \rangle = \langle i | \hat{E}^-(\mathbf{r}, t) \hat{E}^+(\mathbf{r}, t) | i \rangle. \quad (5.13)$$

Here the $\sum_f |f\rangle \langle f| = 1$ relation was used.

Shall us define the first-order correlation function

$$g^{(1)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) \sim \langle i | \hat{E}^-(\mathbf{r}_1, t_1) \hat{E}^+(\mathbf{r}_2, t_2) | i \rangle = \langle \hat{E}^-(\mathbf{r}_1, t_1) \hat{E}^+(\mathbf{r}_2, t_2) \rangle. \quad (5.14)$$

It is useful for quantifying the coherence between two electric fields, as measured in a Michelson or other linear optical interferometer. Here we used the average notation as

$$\langle \hat{A} \rangle \stackrel{\text{def}}{=} \langle i | \hat{A} | i \rangle. \quad (5.15)$$

The detection probability of two photons is given by

$$\begin{aligned} \omega_2(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) &= \sum_f \left| \langle f | \hat{E}^+(\mathbf{r}_1, t_1) \hat{E}^+(\mathbf{r}_2, t_2) | i \rangle \right|^2 = \\ &= \langle i | \hat{E}^-(\mathbf{r}_2, t_2) \hat{E}^-(\mathbf{r}_1, t_1) \hat{E}^+(\mathbf{r}_1, t_1) \hat{E}^+(\mathbf{r}_2, t_2) | i \rangle. \end{aligned} \quad (5.16)$$

The joint probability of photodetection is thus governed by the second-order correlation function:

$$g^{(2)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \frac{\langle \hat{E}^-(\mathbf{r}_2, t_2) \hat{E}^-(\mathbf{r}_1, t_1) \hat{E}^+(\mathbf{r}_1, t_1) \hat{E}^+(\mathbf{r}_2, t_2) \rangle}{\langle \hat{E}^-(\mathbf{r}_2, t_2) \hat{E}^+(\mathbf{r}_2, t_2) \rangle \langle \hat{E}^-(\mathbf{r}_1, t_1) \hat{E}^+(\mathbf{r}_1, t_1) \rangle}. \quad (5.17)$$

It refers to intensity (e.g. for currents). The exact expression for the first-order correlation function is the following:

$$g^{(1)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \frac{\langle \hat{E}^-(\mathbf{r}_1, t_1) \hat{E}^+(\mathbf{r}_2, t_2) \rangle}{\sqrt{\langle \hat{E}^-(\mathbf{r}_2, t_2) \hat{E}^+(\mathbf{r}_2, t_2) \rangle \langle \hat{E}^-(\mathbf{r}_1, t_1) \hat{E}^+(\mathbf{r}_1, t_1) \rangle}}. \quad (5.18)$$

In fact it refers to interference in quantum mechanic.

A particular case is often used, precisely a case $\mathbf{r}_1 = \mathbf{r}_2$ and $\tau = t_2 - t_1$. In such case notation is $g^{(2)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) \stackrel{\text{def}}{=} g^{(2)}(\tau)$. So we have

$$g^{(2)}(\tau) = \frac{\langle \hat{E}^-(\tau) \hat{E}^-(0) \hat{E}^+(0) \hat{E}^+(\tau) \rangle}{\langle \hat{E}^-(0) \hat{E}^+(0) \rangle^2}, \quad (5.19)$$

where $\langle \hat{E}^-(0) \hat{E}^+(0) \rangle = \langle \hat{E}^-(\tau) \hat{E}^+(\tau) \rangle$, $\forall \tau$ was used which means that the signal is homogeneous. For a case of only *one mode field* we can write

$$g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(\tau) \hat{a}^\dagger(0) \hat{a}(0) \hat{a}(\tau) \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2}. \quad (5.20)$$

Important to notice that in numerator of fraction is *normally ordered*.

Example: The second-order correlation function for Fock states ($|i\rangle = |n\rangle$).

$$g^{(2)}(0) = \frac{\langle n | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} | n \rangle}{\langle n | \hat{a}^\dagger \hat{a} | n \rangle} = \frac{n(n-1)}{n^2} = 1 - \frac{1}{n} < 1, \quad \forall n. \quad (5.21)$$

This is a factor of a single-photon beam, e.g. $n = 1 \rightarrow g^{(2)}(0) = 0$. Exactly this effect was observed in a HBT experiment!

Homework. The second-order correlation function for coherent states.

Compute $g^{(2)}(0)$ for case $|i\rangle = |\alpha\rangle$.

6 Atom–field interaction. Quantum approach

6.1 Jaynes–Cummings model (RWA)

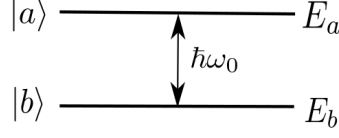


Figure 18: Two-level system

Hamiltonian of system "quantum atom" + "quantum field":

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_A + \hat{\mathcal{H}}_F + \hat{\mathcal{H}}_{int}. \quad (6.1)$$

where $\hat{\mathcal{H}}_A$ describes only *atom*, $\hat{\mathcal{H}}_F$ describes only *quantized field* and $\hat{\mathcal{H}}_{int}$ — interaction part. Explicit expressions are:

$$\hat{\mathcal{H}}_A = E_a |a\rangle \langle a| + E_b |b\rangle \langle b|, \quad (6.2)$$

$$\hat{\mathcal{H}}_F = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} \left(\hat{n}_{\mathbf{k}} + \frac{1}{2} \right), \quad \hat{n}_{\mathbf{k}} = \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}, \quad (6.3)$$

$$\hat{\mathcal{H}}_{int} = -e\mathbf{r} \cdot \hat{\mathbf{E}}, \quad (6.4)$$

where field operator is given by

$$\hat{\mathbf{E}} = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}} \mathbf{e}_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} + \hat{a}_{\mathbf{k}}^\dagger \mathbf{e}_{\mathbf{k}}^* e^{-i\mathbf{k}\mathbf{r}} \right), \quad (6.5)$$

where $\mathbf{e}_{\mathbf{k}}$ is polarization vector. For simplicity we consider case $\mathbf{k} = 0$ and $\mathbf{e}_{\mathbf{k}} \in \mathbb{R}$ so we have

$$\hat{\mathbf{E}} = \sum_{\mathbf{k}} \mathbf{e}_{\mathbf{k}} \varepsilon_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}}^\dagger + \hat{a}_{\mathbf{k}} \right). \quad (6.6)$$

Now let us rewrite dipole moment:

$$e\mathbf{r} = \hat{\mathbf{1}} \cdot e\mathbf{r} \cdot \hat{\mathbf{1}} = \sum_{i,j=a,b} |i\rangle \underbrace{\langle i| e\mathbf{r} |j\rangle}_{\mathbf{d}_{ij}} \langle j| = \sum_{ij} \mathbf{d}_{ij} \underbrace{|i\rangle \langle j|}_{\hat{\sigma}_{ij}} = \sum_{ij} \mathbf{d}_{ij} \hat{\sigma}_{ij}. \quad (6.7)$$

Here we denoted:

$$\mathbf{d}_{ij} \stackrel{\text{def}}{=} \langle i| e\mathbf{r} |j\rangle, \quad \hat{\sigma}_{ij} \stackrel{\text{def}}{=} |i\rangle \langle j|. \quad (6.8)$$

Then

$$\hat{\mathcal{H}}_{int} = - \sum_{ij} \sum_{\mathbf{k}} (\mathbf{d}_{ij} \cdot \mathbf{e}_{\mathbf{k}}) \sigma_{ij} \varepsilon_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}}^\dagger + \hat{a}_{\mathbf{k}} \right). \quad (6.9)$$

Introduce a new notation:

$$\boxed{g_{ij}^{\mathbf{k}} \stackrel{\text{def}}{=} -\frac{\varepsilon_{\mathbf{k}} (\mathbf{d}_{ij} \cdot \mathbf{e}_{\mathbf{k}})}{\hbar}}. \quad (6.10)$$

This is a similarity to Rabi frequency but for one mode of field. To simplify computations let $g_{ij}^{\mathbf{k}} \in \mathbb{R}$. So our Mr. Hamiltonian

$$\hat{\mathcal{H}} = E_a \hat{\sigma}_{aa} + E_b \hat{\sigma}_{bb} + \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{1}{2} \right) + \sum_{ij} \sum_{\mathbf{k}} \hbar g_{ij}^{\mathbf{k}} \hat{\sigma}_{ij} \left(\hat{a}_{\mathbf{k}}^\dagger + \hat{a}_{\mathbf{k}} \right). \quad (6.11)$$

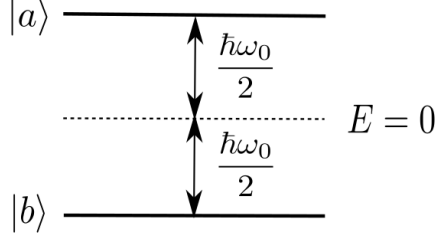


Figure 19: A free choice of initial energy level

Let us put a starting energy point right in between E_a and E_b (fig.19). If we do so then

$$E_a \hat{\sigma}_{aa} + E_b \hat{\sigma}_{bb} = \frac{1}{2} \hbar \omega_0 (\hat{\sigma}_{aa} - \hat{\sigma}_{bb}) + \frac{1}{2} (E_a + E_b) \underbrace{(\hat{\sigma}_{aa} + \hat{\sigma}_{bb})}_{\hookrightarrow = \hat{1}}. \quad (6.12)$$

After that we make a transition to the new Hamiltonian by this energy shift

$$\hat{\mathcal{H}} - \frac{1}{2} (E_a + E_b) \rightarrow \hat{\mathcal{H}}. \quad (6.13)$$

It is convenient to make new denotions:

$$\hat{\sigma}_z \stackrel{\text{def}}{=} \hat{\sigma}_{bb} - \hat{\sigma}_{aa}, \quad (6.14)$$

$$\hat{\sigma}_+ \stackrel{\text{def}}{=} \hat{\sigma}_{ab} = |a\rangle \langle b|, \quad (6.15)$$

$$\hat{\sigma}_- \stackrel{\text{def}}{=} \hat{\sigma}_{ba} = |b\rangle \langle a|, \quad (6.16)$$

where

$$|b\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |a\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (6.17)$$

so reader may easily construct matrices for $\hat{\sigma}_z$, $\hat{\sigma}_+$ and $\hat{\sigma}_-$:

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\sigma}_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (6.18)$$

To answer how act new operators $\hat{\sigma}_+$ and $\hat{\sigma}_-$ consider the following:

$$\hat{\sigma}_+ |b\rangle = |a\rangle \langle b|b\rangle = |a\rangle, \quad (6.19)$$

$$\hat{\sigma}_- |a\rangle = |b\rangle \langle a|a\rangle = |b\rangle. \quad (6.20)$$

The operator $\hat{\sigma}_+$ takes the system from lower state to upper states vice versa.

Usually $\mathbf{d}_{aa} = \mathbf{d}_{bb} = 0$, so

$$g_{aa}^{\mathbf{k}} = 0, \quad g_{bb}^{\mathbf{k}} = 0, \quad (6.21)$$

$$g_{ab}^{\mathbf{k}} = g_{ba}^{\mathbf{k}} \stackrel{\text{def}}{=} g^{\mathbf{k}}. \quad (6.22)$$

After that Hamiltonian may be written as

$$\hat{\mathcal{H}} = \underbrace{\sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{1}{2} \right)}_{\text{field}} + \underbrace{\frac{\hbar \omega_0}{2} \hat{\sigma}_z}_{\text{atom}} + \underbrace{\hbar \sum_{\mathbf{k}} g^{\mathbf{k}} (\hat{\sigma}_+ + \hat{\sigma}_-) \left(\hat{a}_{\mathbf{k}}^\dagger + \hat{a}_{\mathbf{k}} \right)}_{\text{interaction}}. \quad (6.23)$$

Algebra of new operators:

$$[\hat{\sigma}_-, \hat{\sigma}_+] = -\hat{\sigma}_z, \quad (6.24)$$

$$[\hat{\sigma}_-, \hat{\sigma}_z] = 2\hat{\sigma}_-, \quad (6.25)$$

$$[\hat{\sigma}_+, \hat{\sigma}_z] = -2\hat{\sigma}_+, \quad (6.26)$$

$$\{\hat{\sigma}_+, \hat{\sigma}_-\} = \hat{\mathbb{1}}. \quad (6.27)$$

Last property is rooted in fact that electrons are fermions. $\hat{\sigma}_+$ and $\hat{\sigma}_-$ are creation and annihilation operators for electron in atom. Lets consider the physical meaning of the interactive terms:

$$(\hat{\sigma}_+ + \hat{\sigma}_-) (\hat{a}_\mathbf{k}^\dagger + \hat{a}_\mathbf{k}) = \underbrace{\hat{\sigma}_+ \hat{a}}_{(I)} + \underbrace{\hat{\sigma}_+ \hat{a}^\dagger}_{\text{unphysical (II)}} + \underbrace{\hat{\sigma}_- \hat{a}}_{\text{unphysical (III)}} + \underbrace{\hat{\sigma}_- \hat{a}^\dagger}_{(IV)}, \quad (6.28)$$

where (I) — photon annihilation and electron excitation, (II) — photon creation and electron excitation, (III) — photon annihilation and electron relaxation, (IV) — photon creation and electron relaxation. Terms $\hat{\sigma}_+ \hat{a}^\dagger + \hat{\sigma}_- \hat{a}$ are unphysical, so *may be omitted*. In fact this is the RWA. After such assumption we have:

$$\boxed{\hat{\mathcal{H}} = \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} \left(\hat{a}_\mathbf{k}^\dagger \hat{a}_\mathbf{k} + \frac{1}{2} \right) + \frac{\hbar \omega_0}{2} \hat{\sigma}_z + \sum_{\mathbf{k}} \hbar g_{\mathbf{k}} \left(\hat{\sigma}_+ \hat{a}_\mathbf{k} + \hat{a}_\mathbf{k}^\dagger \hat{\sigma}_- \right)}. \quad (6.29)$$

For simplicity let us consider one mode field

$$\hat{\mathcal{H}} = \underbrace{\hbar \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)}_{\hat{\mathcal{H}}_0} + \underbrace{\hbar g (\hat{\sigma}_+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_-)}_{\hat{V}}. \quad (6.30)$$

Now we pass into the representation of interaction

$$\hat{\mathcal{H}}_V = e^{\frac{i}{\hbar} \hat{\mathcal{H}}_0 t} \hat{V} e^{-\frac{i}{\hbar} \hat{\mathcal{H}}_0 t}. \quad (6.31)$$

Remark: The interaction representation means the following. After unitary transformation

$$|\tilde{\psi}\rangle = e^{\frac{i}{\hbar} \hat{\mathcal{H}}_0 t} |\psi\rangle, \quad \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}_V$$

we obtain new effective Schrödinger equation

$$i\hbar |\dot{\psi}\rangle = \hat{\mathcal{H}} |\psi\rangle \rightarrow i\hbar |\dot{\tilde{\psi}}\rangle = \hat{\mathcal{H}}_V |\tilde{\psi}\rangle.$$

Let us begin. We need to consider

$$e^{i\frac{\omega_0}{2}\hat{\sigma}_z t} e^{i\omega\hat{a}^\dagger\hat{a}t} \hat{V} e^{-i\omega\hat{a}^\dagger\hat{a}t} e^{-i\frac{\omega_0}{2}\hat{\sigma}_z t}. \quad (6.32)$$

Different terms will appear:

$$e^{i\omega\hat{a}^\dagger\hat{a}t} \hat{a} e^{-i\omega\hat{a}^\dagger\hat{a}t} = \hat{a} e^{-i\omega t}, \quad (6.33)$$

$$e^{i\omega\hat{a}^\dagger\hat{a}t} \hat{a}^\dagger e^{-i\omega\hat{a}^\dagger\hat{a}t} = \hat{a}^\dagger e^{i\omega t}, \quad (6.34)$$

$$e^{i\frac{\omega_0}{2}\hat{\sigma}_z t} \hat{\sigma}_+ e^{-i\frac{\omega_0}{2}\hat{\sigma}_z t} = \hat{\sigma}_+ e^{i\omega_0 t}, \quad (6.35)$$

$$e^{i\frac{\omega_0}{2}\hat{\sigma}_z t} \hat{\sigma}_- e^{-i\frac{\omega_0}{2}\hat{\sigma}_z t} = \hat{\sigma}_- e^{-i\omega_0 t}. \quad (6.36)$$

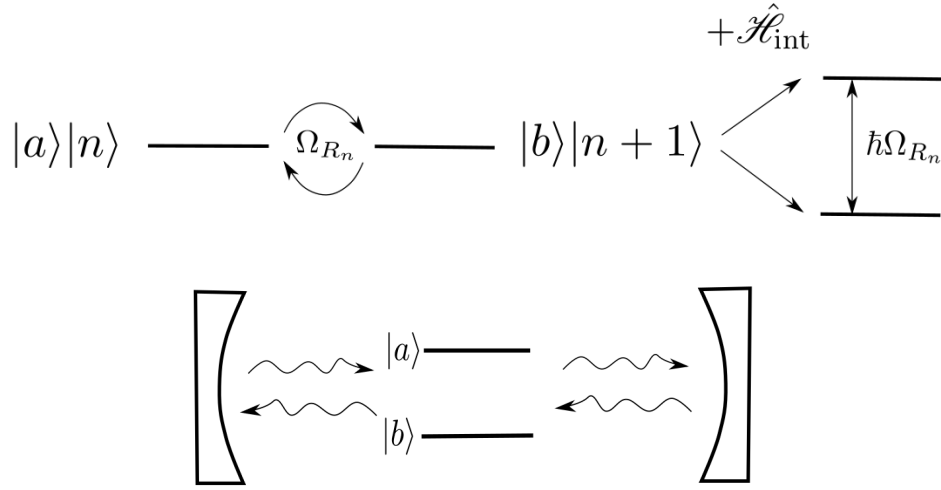


Figure 20: Oscillations in a cavity

So

$$\hat{\mathcal{H}}_V = \hbar g \left(\hat{a}^\dagger \hat{\sigma}_- e^{-i(\omega_0 - \omega)t} + \hat{\sigma}_+ \hat{a} e^{i(\omega_0 - \omega)t} \right). \quad (6.37)$$

As a matter of convenience we introduce $\Delta \stackrel{\text{def}}{=} \omega_0 - \omega$. An effective Schrödinger equation now may be written as

$$i\hbar \frac{\partial}{\partial t} |\tilde{\psi}\rangle = \hat{\mathcal{H}}_V |\tilde{\psi}\rangle. \quad (6.38)$$

After that we expand wave function in series

$$|\tilde{\psi}\rangle = \sum_n C_{a,n}(t) |n\rangle |a\rangle + \sum_n C_{b,n}(t) |n\rangle |b\rangle, \quad (6.39)$$

where $C_{a,n}(t)$ — a probability of finding an electron in $|a\rangle$ and a photon in $|n\rangle$ at time t . To bring into focus results we omit all the computations and present only the solution:

$$\begin{cases} i\hbar \dot{C}_{a,n}(t) = \hbar g \sqrt{n+1} C_{b,n+1}(t) e^{i\Delta t}, \\ i\hbar \dot{C}_{b,n+1}(t) = \hbar g \sqrt{n+1} C_{a,n}(t) e^{-i\Delta t}. \end{cases} \quad (6.40)$$

For simplicity let us consider a resonant excitation, it means we need to put $\Delta = 0$ in (6.40). After we have

$$\begin{cases} \dot{C}_{a,n}(t) = -ig\sqrt{n+1} C_{b,n+1}(t) \\ \dot{C}_{b,n+1}(t) = -ig\sqrt{n+1} C_{a,n}(t) \end{cases} \rightarrow \ddot{C}_{a,n}(t) + \underbrace{g^2(n+1)}_{\Omega_{R_n}^2} C_{a,n}(t) = 0. \quad (6.41)$$

Easy to notice, we obtained oscillations with angular frequency

$$\boxed{\Omega_{R_n} = g\sqrt{n+1}}. \quad (6.42)$$

A physical interpretation of such oscillations is shown in fig. 20.

Example: Vacuum Rabi oscillations.

If we put $n = 0$ we get $\Omega_{R_0} = g \neq 0$. We have a vacuum interaction: no field, interaction exists!

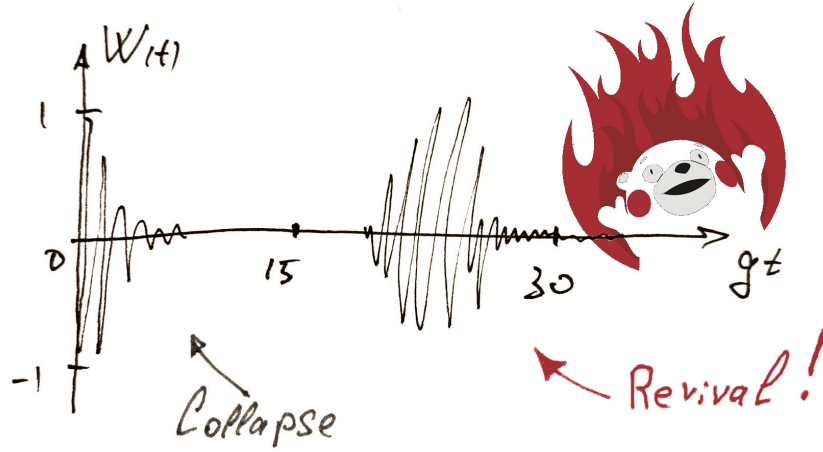


Figure 21: Time evolution of the population inversion $W(t)$ for an initially coherent state.

Homework. Deadline: 29th of December

1. (6 pts) (See problem 6.2 from Scully). One of the common models of light-atom interaction in a cavity can be described by the Hamiltonian :

$$H = \hbar\omega_0\sigma_z + \hbar\omega a^\dagger a + \hbar g(\sqrt{a^\dagger a} a^\dagger \sigma_- + \sigma_+ a \sqrt{a^\dagger a}),$$

where interaction depends on the intensity. Compute the inverse population at the timescale of Rabi oscillations, considering initial state of photons as coherent.

6.2 Collapse and revival

Another important quantity is the inversion $W(t)$ which is related to the probability amplitudes $C_{a,n}(t)$ and $C_{b,n}(t)$ by the expression

$$W(t) = \sum_n (|C_{a,n}(t)|^2 - |C_{b,n}(t)|^2). \quad (6.43)$$

In fig. 21 $W(t)$ is plotted as a function of normalized time gt for an initial coherent state. The behavior of $W(t)$ is quite different from the corresponding curve (fig. 2) in the semiclassical theory. In the present case the envelope of the sinusoidal Rabi oscillations 'collapses' to zero. However as time increases we encounter a 'revival' of the collapsed inversion. This behavior of collapse and revival of inversion is repeated with increasing time, with the amplitude of Rabi oscillations decreasing and the time duration in which revival takes place increasing and ultimately overlapping with the earlier revival.

6.3 Energy spectrum. Dispersion relation

Let us find eigenstates of $\hat{\mathcal{H}}$ which is given by (6.30). Here, for convenience, the vacuum field energy is set to 0, so

$$\hat{\mathcal{H}} = \underbrace{\hbar\omega\hat{n} + \frac{\hbar\omega_0}{2}\hat{\sigma}_z}_{\hat{\mathcal{H}}_0} + \underbrace{\hbar g(\hat{\sigma}_+\hat{a} + \hat{a}^\dagger\hat{\sigma}_-)}_{\hat{\mathcal{H}}_{\text{int}}}. \quad (6.44)$$

The equation for eigenvalues is

$$\hat{\mathcal{H}}|\psi\rangle = E|\psi\rangle. \quad (6.45)$$

Next terms will appear:

$$\hat{\mathcal{H}}_0 |a\rangle |n\rangle = \left(\frac{\hbar\omega_0}{2} + \hbar\omega n \right) |a\rangle |n\rangle, \quad (6.46)$$

$$\hat{\mathcal{H}}_0 |b\rangle |n+1\rangle = \left(-\frac{\hbar\omega_0}{2} + \hbar\omega n \right) |b\rangle |n+1\rangle, \quad (6.47)$$

$$\hat{\mathcal{H}}_{\text{int}} |a\rangle |n\rangle = \hbar g \sqrt{n+1} |b\rangle |n+1\rangle, \quad (6.48)$$

$$\hat{\mathcal{H}}_{\text{int}} |b\rangle |n+1\rangle = \hbar g \sqrt{n+1} |a\rangle |n\rangle. \quad (6.49)$$

Important to notice that number of quantas is concerned, if we take basis $|a\rangle |n\rangle$ and $|b\rangle |n+1\rangle$. After acting $\hat{\mathcal{H}}_{\text{int}}$ we stay in the same subspace with the same basis. Consequently, eigenfunctions may be written as

$$|\psi\rangle = |\varphi_1\rangle \cdot |\varphi_2\rangle \cdot \dots \cdot |\varphi_n\rangle \cdot \dots \quad (6.50)$$

Each of $|\varphi_n\rangle$ acts on its own subspace and does not affect the others:

$$|\varphi_n\rangle = c_n |a\rangle |n\rangle + c_{2n} |b\rangle |n+1\rangle. \quad (6.51)$$

It leads to a very imports consequence — $\hat{\mathcal{H}}$ is *block-diagonal matrix*:

$$\hat{\mathcal{H}} = \begin{pmatrix} \hat{\mathcal{H}}_1 & 0 & 0 & \dots & 0 & \dots \\ 0 & \hat{\mathcal{H}}_2 & 0 & \dots & 0 & \dots \\ \vdots & \dots & \ddots & \dots & \vdots & \dots \\ 0 & \dots & 0 & \hat{\mathcal{H}}_n & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \quad (6.52)$$

Where each of $\hat{\mathcal{H}}_n$ is a 2×2 matrix.

Looking for an energy spectrum (6.45) is equivalent to the Hamiltonian diagonalization. We simplified our problem and now we need only to diagonalize 2×2 matrices or in other words we need to solve

$$\hat{\mathcal{H}}_n |\varphi_n\rangle = E_n |\varphi_n\rangle. \quad (6.53)$$

In an explicit form we need to diagonalize

$$\begin{aligned} \hat{\mathcal{H}}_n &= \hat{\mathcal{H}}_n^0 + \hat{\mathcal{H}}_n^{\text{int}} = \begin{pmatrix} \frac{\hbar\omega_0}{2} + \hbar\omega n & 0 \\ 0 & -\frac{\hbar\omega_0}{2} + \hbar\omega n \end{pmatrix} + \begin{pmatrix} 0 & \hbar g \sqrt{n+1} \\ \hbar g \sqrt{n+1} & 0 \end{pmatrix} = \\ &= \hbar\omega \left(n + \frac{1}{2} \right) \hat{I} + \frac{\hbar}{2} \begin{pmatrix} \Delta & 2g\sqrt{n+1} \\ 2g\sqrt{n+1} & -\Delta \end{pmatrix}, \end{aligned} \quad (6.54)$$

where $\Delta = \omega - \omega_0$ and $\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an identity matrix. Human by nature is lazy, so we do. It means let us consider a simple case when $\Delta = 0$, so

$$\det \left(\hat{\mathcal{H}}_n - E \hat{I} \right) = 0 \quad \rightarrow \quad \left(E - \hbar\omega \left(n + \frac{1}{2} \right) \right) = \hbar^2 g^2 (n+1) \quad (6.55)$$

or

$$\begin{cases} E_1 = \hbar\omega \left(n + \frac{1}{2} \right) + \hbar g \sqrt{n+1}, \\ E_2 = \hbar\omega \left(n + \frac{1}{2} \right) - \hbar g \sqrt{n+1}. \end{cases} \quad (6.56)$$

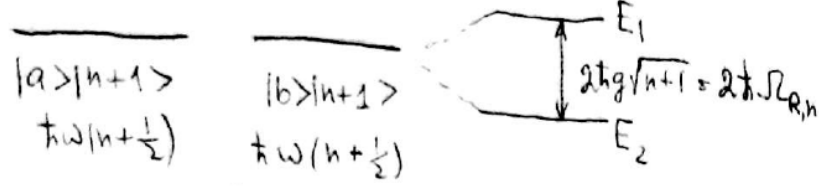


Figure 22: Level splitting after acting $\hat{\mathcal{H}}_n$

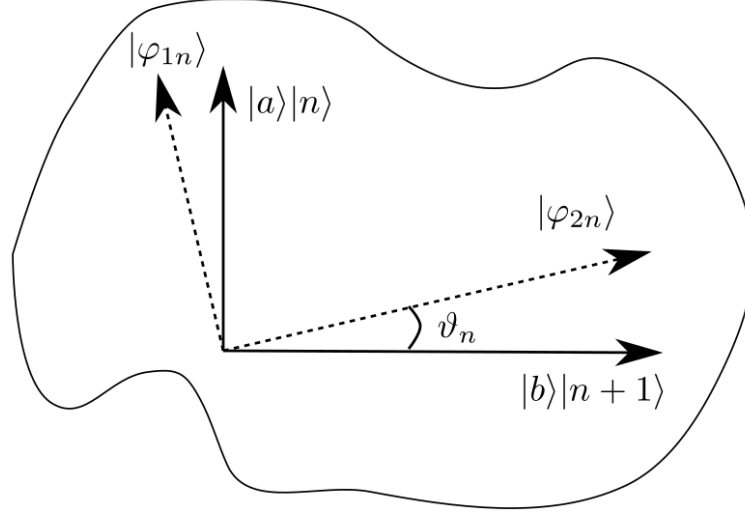


Figure 23: Rotation in a subspace

This energy splitting is shown in fig. 22. If we consider case $\Delta \neq 0$ then we will get

$$\begin{cases} E_1 = \hbar\omega \left(n + \frac{1}{2}\right) + \frac{1}{2}\hbar R_n, \\ E_2 = \hbar\omega \left(n + \frac{1}{2}\right) - \frac{1}{2}\hbar R_n, \end{cases} \quad (6.57)$$

where $R_n = \sqrt{\Delta^2 + 4g^2(n+1)}$.

Eigenstates may be written as a rotated $|a, n\rangle$ and $|b, n+1\rangle$ states

$$\begin{cases} |\varphi_{1n}\rangle = \cos \vartheta_n |a, n\rangle + \sin \vartheta_n |b, n+1\rangle, \\ |\varphi_{2n}\rangle = -\sin \vartheta_n |a, n\rangle + \cos \vartheta_n |b, n+1\rangle, \end{cases} \quad (6.58)$$

where $\cos \vartheta_n = \frac{2g\sqrt{n+1}}{\sqrt{(R_n - \Delta)^2 + 4g^2(n+1)}}$. Just to make things clear, $|a, n\rangle$ and $|b, n+1\rangle$ are eigenfunctions of $\hat{\mathcal{H}}_n^0$, after 'turning on' an interaction we gain a plane rotation!

States $|\varphi_{1n}\rangle$ and $|\varphi_{2n}\rangle$ — *polariton states*. This is a mixed state of light and medium.

For $\Delta = 0$ we have $\vartheta = \frac{\pi}{4}$. It means that

$$\begin{cases} |\varphi_{1n}\rangle = \frac{1}{\sqrt{2}} (|a, n\rangle + |b, n+1\rangle), \\ |\varphi_{2n}\rangle = \frac{1}{\sqrt{2}} (-|a, n\rangle + |b, n+1\rangle). \end{cases} \quad (6.59)$$

Energy is given by (6.57). If there is no interaction ($g = 0$) we have

$$\begin{cases} E_{1n} = \hbar\omega \left(n + \frac{1}{2}\right) + \frac{\hbar}{2}\Delta, \\ E_{2n} = \hbar\omega \left(n + \frac{1}{2}\right) - \frac{\hbar}{2}\Delta. \end{cases} \quad (6.60)$$

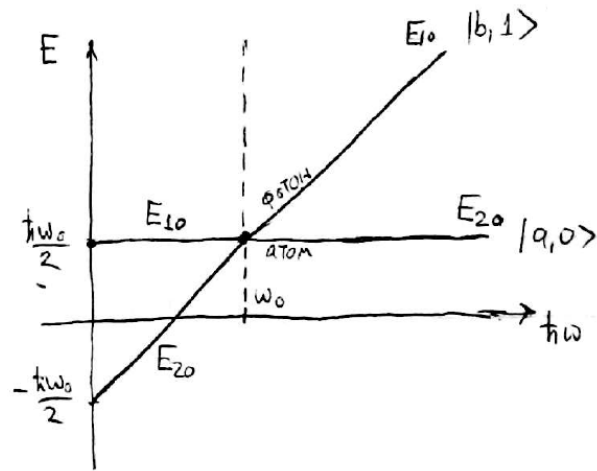


Figure 24: Case of $n = 0$ and $g = 0$. State $|b, 1\rangle$ — there is a photon with energy $\hbar\omega$, so $E = E(\omega)$ is linear. State $|a, 0\rangle$ — no photon in cavity, energy does not depend on $\hbar\omega$, there is an exciton

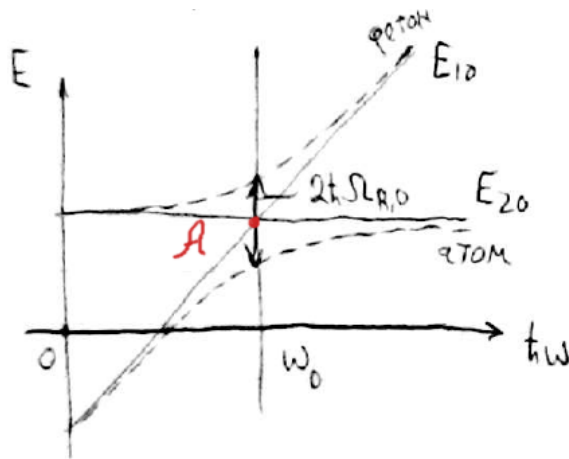


Figure 25: Case of $n = 0$ and $g \neq 0$. Line splitting is observed. At frequency ω_0 there is no pure atom state nor photon state — it is mixed state, polariton state

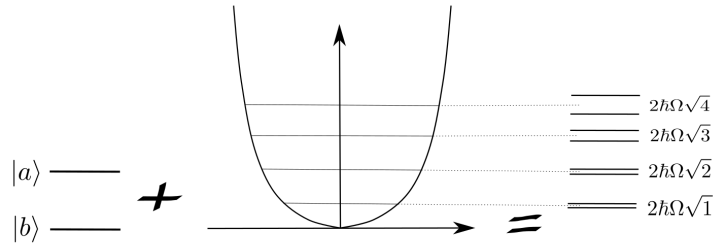


Figure 26: Cummings ladder. Energy splitting as a result of combining two quantum systems: atom and field

Not let us make a graphical analysis. At first shall we put $n = 0$ and $g = 0$ (fig. 24). After that we 'turn on' interaction ($n = 0, g \neq 0$) and we get fig. 25. All necessary comments are given under the figures.

At point A on fig. 25 quantum of energy neither in atom nor in photon, it is equidistant from E_{01} and E_{02} . This is a *polariton state*!

In fact we described a so called Cummings ladder (fig. 26).

7 Spontaneous relaxation. Weisskopf-Wigner theory

We have shown that quantum approach to atom-field interaction results in Rabi oscillation between the excited atomic state and photon states. In particular, the interaction with field vacuum results in vacuum Rabi oscillations and solves the problem of atomic state evolution in the absence of the excitation field. However, the the origin of spontaneous emission of excited stated has not been discussed yet. At the same time, we have seen that effects similar to exponential decay of the excited state appears in collapse-revival dynamics, when an atom interacts with many photon states of a *single mode field*. In this lecture, we will see that spontaneous emission originates from interaction with *multimode field*.

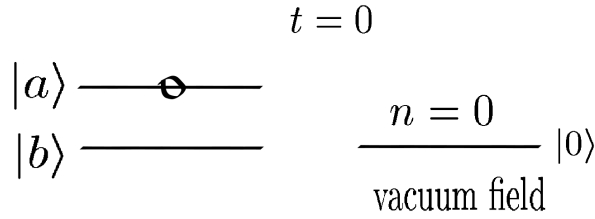


Figure 27: Initial conditions

As in the previous lecture, we start from the interaction Hamiltonian in the interaction picture and within RWA:

$$\hat{V}(t) = \hbar \sum_{\mathbf{k}} [g_{\mathbf{k}}^* \hat{\sigma}_+ \hat{a}_{\mathbf{k}} e^{i(\omega_0 - \omega_{\mathbf{k}})t} + H.c.] . \quad (7.1)$$

In order to study the dynamics of the system, we write down the wavefunction of the system with time dependent coefficients:

$$|\psi(t)\rangle = \sum_{n, \mathbf{k}} (c_{a, n_{\mathbf{k}}}(t) |a\rangle |n_{\mathbf{k}}\rangle + c_{b, n_{\mathbf{k}}}(t) |b\rangle |n_{\mathbf{k}}\rangle) , \quad (7.2)$$

where $c_{a(b), n_{\mathbf{k}}} \stackrel{\text{def}}{=} c_{a(b), n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, \dots}$ and $|n_{\mathbf{k}}\rangle \stackrel{\text{def}}{=} |n_{\mathbf{k}_1}\rangle |n_{\mathbf{k}_2}\rangle \cdot \dots$. We assume that at time $t = 0$ the atom is in the excited state $|a\rangle$ and the field modes are in the vacuum state $|0\rangle$. That means that one needs to consider only state with total number of quanta equal to one. That reduces the wavefunction to the following form:

$$|\psi(t)\rangle = c_{a,0}(t) |a\rangle |0\rangle + \sum_{\mathbf{k}} c_{b, \mathbf{k}}(t) |b\rangle |1_{\mathbf{k}}\rangle , \quad (7.3)$$

which should satisfy the Schroedinger equation

$$i\hbar \left| \dot{\psi}(t) \right\rangle = \hat{V}(t) |\psi(t)\rangle \quad (7.4)$$

supported by the initial conditions

$$c_{a,0}(0) = 1, \quad c_{b, \mathbf{k}}(0) = 0. \quad (7.5)$$

From the Schroedinger equation we get the equations of motion for the probability amplitudes:

$$\begin{cases} \dot{c}_{a,0}(t) = -i \sum_{\mathbf{k}} g_{\mathbf{k}}^* e^{i\Delta_{\mathbf{k}} t} c_{b,1_{\mathbf{k}}}(t), \\ \dot{c}_{b,1_{\mathbf{k}}}(t) = -i g_{\mathbf{k}} e^{-i\Delta_{\mathbf{k}} t} c_{a,0}(t), \end{cases} \quad (7.6a)$$

$$\quad (7.6b)$$

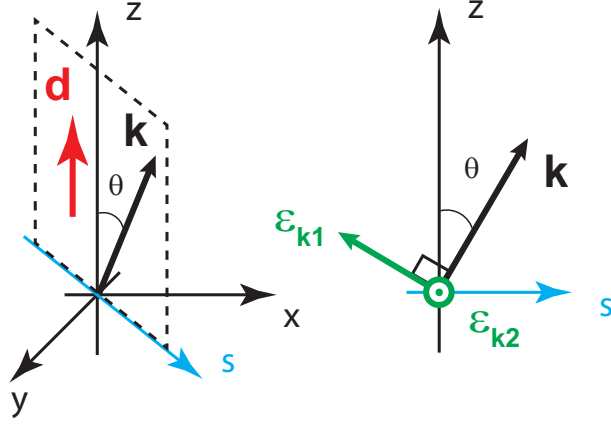


Figure 28: Illustration of k -space integration

where $\Delta_{\mathbf{k}} \stackrel{\text{def}}{=} \omega_0 - \omega_{\mathbf{k}}$. To solve this system, firstly, we integrate the equation $\int (7.6b) dt$:

$$c_{b,1\mathbf{k}}(t) = -ig_{\mathbf{k}} \int_0^t d\tau e^{-i\Delta_{\mathbf{k}}\tau} c_{a,0}(\tau). \quad (7.7)$$

After that, we substitute (7.7) to (7.6a) and get the differential-integral equation

$$\dot{c}_{a,0}(t) = -\sum_{\mathbf{k}} |g_{\mathbf{k}}|^2 \int_0^t d\tau e^{-i\Delta_{\mathbf{k}}(\tau-t)} c_{a,0}(\tau). \quad (7.8)$$

This is still an exact equation. However, its solution is a complicated mathematical problem, and we make some approximations. Assuming that the modes of the field are closely spaced in k -space, we can

$$\sum_{\mathbf{k}} \xrightarrow{L \rightarrow \infty} \sum_{\lambda} V \int \frac{d^3k}{(2\pi)^3} \quad (7.9)$$

where we have left summation over different polarizations, and $\int d^3k = \int_0^\infty k^2 dk \int_0^{2\pi} d\varphi \int_0^\pi \sin\vartheta d\vartheta$ in the spherical coordinate system. We recall that $g_{\mathbf{k},\lambda} = -(\mathbf{d} \cdot \boldsymbol{\varepsilon}_{\mathbf{k},\lambda})/\hbar$ is the coupling constant between the atom and $\mathbf{k}$ -mode with λ polarization. System has a preferred direction and we direct the z -axis along the dipole moment $\mathbf{d}$. For each k -vector we fix one of the polarizations laying in the plane containing z -axis and k -vector as shown in Fig. 28, while another polarization vector becomes orthogonal to z -axis and dipole moment $\mathbf{d}$. The one can account for interaction with one mode only:

$$g_{\mathbf{k},1} = -\frac{|\mathbf{d}| \varepsilon_{\mathbf{k},1}}{\hbar} \sin\vartheta. \quad (7.10)$$

This allows us easily integrate over ϑ and φ :

$$\int_0^{2\pi} \int_0^\pi |g_{\mathbf{k},1}|^2 \sin\vartheta d\varphi d\vartheta = \frac{(|\mathbf{d}| \varepsilon_{\mathbf{k},1})^2}{\hbar^2} \cdot 2\pi \int_0^\pi d\vartheta \sin^3\vartheta = \frac{(|\mathbf{d}| \varepsilon_{\mathbf{k},1})^2}{\hbar^2} \frac{4\pi}{3}. \quad (7.11)$$

It means that (7.8) becomes

$$\dot{c}_{a,0}(t) = -\frac{2V}{(2\pi\hbar)^3} \cdot \frac{4\pi}{3} |\mathbf{d}|^2 \int_0^\infty dk \int_0^t d\tau k^2 \varepsilon_{\mathbf{k},1}^2 e^{-i\Delta_{\mathbf{k}}(\tau-t)} c_{a,0}(\tau). \quad (7.12)$$

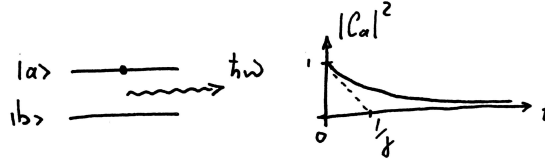


Figure 29: Relaxation process

Now, we rewrite the $\int dk$ part using dispersion law $k = \omega_{\mathbf{k}}/c$ and recalling that $\varepsilon_{\mathbf{k}}^2 = \frac{\hbar\omega_{\mathbf{k}}}{2\varepsilon_0 V}$ (see Lecture on secondary quantization):

$$\int_0^\infty dk k^2 \frac{\hbar\omega_{\mathbf{k}}}{2\varepsilon_0 V} e^{-i\Delta_{\mathbf{k}}(\tau-t)} = \frac{\hbar}{2c^3\varepsilon_0 V} \int_0^\infty d\omega_{\mathbf{k}} \omega_{\mathbf{k}}^3 e^{-i(\omega_0-\omega_{\mathbf{k}})(\tau-t)}, \quad (7.13)$$

so

$$\dot{c}_{a,0}(t) = -\frac{2V}{(2\pi\hbar)^3} \cdot \frac{4\pi}{3} |\mathbf{d}|^2 \cdot \frac{\hbar}{2c^3\varepsilon_0 V} \int_0^\infty d\omega_{\mathbf{k}} \int_0^t d\tau \omega_{\mathbf{k}}^3 e^{-i(\omega_0-\omega_{\mathbf{k}})(\tau-t)} c_{a,0}(\tau). \quad (7.14)$$

In the emission spectrum, the intensity of light associated with the emitted radiation is going to be centered around the atomic transition frequency ω_0 . The quantity $\omega_{\mathbf{k}}^3$ varies slowly around $\omega_{\mathbf{k}} = \omega_0$. Therefore here we can use the *stationary-phase method*:

$$\int_0^\infty d\omega_{\mathbf{k}} \omega_{\mathbf{k}}^3 e^{-i(\omega_0-\omega_{\mathbf{k}})(\tau-t)} \rightarrow \omega_0^3 \int_{-\infty}^\infty d\omega_{\mathbf{k}} e^{-i(\omega_0-\omega_{\mathbf{k}})(\tau-t)} = \omega_0^3 \cdot 2\pi\delta(\tau-t). \quad (7.15)$$

Now integral over $d\tau$ is easy to compute $\int_0^t d\tau \delta(\tau-t) c_{a,0}(\tau) = \frac{1}{2} c_{a,0}(t)$, and we finally obtain

$$\dot{c}_{a,0}(t) = -\frac{\gamma_0}{2} c_{a,0}(t), \quad \rightarrow \quad c_{a,0}(t) = \exp\left[-\frac{\gamma_0}{2} t\right], \quad (7.16)$$

where the *free space decay constant*

$$\gamma_0 \stackrel{\text{def}}{=} \frac{\omega_0^3 |\mathbf{d}|^2}{3\pi\hbar c^3 \varepsilon_0}. \quad (7.17)$$

To conclude this part, one can see that interaction of a single excited atom with continuum of states results in the irreversible decay of the excitation into the photon

NB:

- If we consider a medium with refractive index n , which effectively changes the wave vector $k \rightarrow nk$, then we get

$$\gamma_0 = \frac{\omega_0^3 |\mathbf{d}|^2 n}{3\pi\hbar c^3 \varepsilon_0}. \quad (7.18)$$

The origin of this will be also discussed in the following lectures.

- It is useful to remember that $\gamma \sim n|\mathbf{d}|^2$ and expression for γ contains the Plank constant $\hbar$, which means that this process has a quantum nature.

- Substitution of summation over continuum with the integration

$$\sum_{\mathbf{k}} \rightarrow \int d\omega_{\mathbf{k}} \rightarrow \delta(t - \tau) \quad (7.19)$$

is called the Weisskopf-Wigner method.

8 Dipole radiation. Dyadic Green's function. The Purcell effect: classical approach

8.1 Dipole radiation and dyadic Green's function

NB: Brief overview of Green's functions

A Green's function, $G(x, s)$, of a linear differential operator $\hat{L}$ acting on distributions over a subset of the Euclidean space, at a point s , is any solution of

$$\hat{L}G(x, s) = \delta(x - s). \quad (8.1)$$

The knowledge of $G(x, s)$ allows one to write the solution of the differential equation with arbitrary inhomogeneous function in rhs. In other words,

$$\hat{L}u(x) = f(x) \quad \rightarrow \quad u(x) = \int ds G(x, s) f(s). \quad (8.2)$$

Consider an oscillating point dipole $\mathbf{d}(t) = \mathbf{d}_0 e^{-i\omega t}$ which is located in the point $\mathbf{r}_0$. Let us find the electric field generated by this dipole in a free space. In order to solve the Maxwell equation we express the dipole source in terms of current density a time derivative of polarization density:

$$\mathbf{j} = \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial \mathbf{d}(t)}{\partial t} \delta(\mathbf{r} - \mathbf{r}_0) = -i\omega \mathbf{d}(t) \delta(\mathbf{r} - \mathbf{r}_0) \quad (8.3)$$

We then can write down the Maxwell equations with account for the currents sources (in SI units)

$$\left\{ \begin{array}{l} \text{rot } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \end{array} \right. \quad (8.4a)$$

$$\left\{ \begin{array}{l} \text{rot } \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, \end{array} \right. \quad (8.4b)$$

$$\left\{ \begin{array}{l} \text{div } \mathbf{D} = \rho, \end{array} \right. \quad (8.4c)$$

$$\left\{ \begin{array}{l} \text{div } \mathbf{B} = 0. \end{array} \right. \quad (8.4d)$$

After applying the Fourier transform over time (in fact just $\partial_t \rightarrow -i\omega$) and using $\mathbf{D} = \varepsilon \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu \mu_0 \mathbf{H}$ we obtain

$$\text{rot rot } \mathbf{E} - \mu \mu_0 \varepsilon \varepsilon_0 \omega^2 \mathbf{E} = i\mu \mu_0 \omega \mathbf{j} \quad (8.5)$$

introducing $k = \frac{\omega}{c} \sqrt{\varepsilon \mu}$ we get

$$\text{rot rot } \mathbf{E} - k^2 \mathbf{E} = \frac{k^2}{\varepsilon \varepsilon_0} \mathbf{d}_0 \delta(\mathbf{r} - \mathbf{r}_0). \quad (8.6)$$

Here we have an inhomogeneous differential equation with a δ -function on the r.h.s. of the equation. Its solution manifests itself in a Green's function $\hat{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0)$, however as Eq. (8.6) is a vectorial equation then the Green's function for electromagnetic field is a tensor of second rank or a so-called *dyadic Green's function*. This fact is represented by writing a "hat" over G . By having the Green's function (the radiation of a arbitrary oriented dipole), we can calculate radiation for any complex current $\mathbf{j}(\mathbf{r})$.

Example: Field from an arbitrary current

$$\mathbf{E}(\mathbf{r}) = i \begin{pmatrix} \frac{4\pi}{c} \\ \frac{1}{c\varepsilon_0} \end{pmatrix} k \int d^3r' \hat{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \mathbf{j}(\mathbf{r}'), \quad \text{units: } \begin{pmatrix} \text{sgs} \\ \text{SI} \end{pmatrix} \quad (8.7)$$

Remark: $\hat{\mathbf{G}}\mathbf{j} = \mathbf{e}_i \hat{G}_{ik} j_k$, where $\mathbf{e}_i$ is the unit vector.

8.1.1 Derivation of the Green's function for Maxwell equations

We construct now the explicit form of $\hat{\mathbf{G}}$ in a homogeneous medium. The general definition of the dyadic Green's function is defined as a solution of the equation

$$\text{rot rot } \hat{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0) - k^2 \hat{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0) = \hat{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}_0). \quad (8.8)$$

All we need to find is the relation which connects electric field $\mathbf{E}(\mathbf{r}, \omega)$ and current density $\mathbf{j}(\mathbf{r}, \omega)$. In order to find it, one can write down the expression of electric field in terms of scalar and vectorial potentials

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi. \quad (8.9)$$

If we use electromagnetic potential $\mathbf{A}$ then we can choose any gauge for our convenience. Let us take the Lorenz gauge

$$\text{div } \mathbf{A} + \frac{\varepsilon\mu}{c^2} \frac{\partial \varphi}{\partial t} = 0 \quad \rightarrow \quad \nabla \varphi = \frac{c^2}{i\omega\varepsilon\mu} \nabla \text{div } \mathbf{A}. \quad (8.10)$$

By using the gauge formula and relation for vector potential $\mathbf{B} = \text{rot } \mathbf{A}$, one gets the equation for φ and $\mathbf{A}$:

$$\begin{cases} \Delta \mathbf{A} + k^2 \mathbf{A} = -\mu\mu_0 \mathbf{j}, \\ \Delta \varphi + k^2 \varphi = -\frac{\rho}{\varepsilon\varepsilon_0}. \end{cases} \quad \begin{matrix} (8.11a) \\ (8.11b) \end{matrix}$$

As we can see (8.11) — inhomogeneous Helmholtz equations. The Green's function is well-known from classical electrodynamics course:

$$G_0(\mathbf{r}, \mathbf{r}_0) = \frac{e^{ikR}}{4\pi R}, \quad R = |\mathbf{r} - \mathbf{r}_0|. \quad (8.12)$$

It means that we can write the solution of (8.11) as

$$\begin{cases} \mathbf{A}(\mathbf{r}) = \mu\mu_0 \int d^3r' G_0(\mathbf{r}, \mathbf{r}') \hat{\mathbf{I}} \mathbf{j}(\mathbf{r}'), \\ \varphi(\mathbf{r}) = \frac{1}{\varepsilon\varepsilon_0} \int d^3r' G_0(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}'), \end{cases} \quad \begin{matrix} (8.13a) \\ (8.13b) \end{matrix}$$

where $\hat{\mathbf{I}}_{ik} = \delta_{ik}$ — a unit tensor. To get the final answer for $\hat{\mathbf{G}}$ we substitute (8.13), (8.10) in (8.9) and compare the result with (8.7). Substitution gives

$$\mathbf{E}(\mathbf{r}) = i\omega \mathbf{A} - \frac{c^2}{i\omega\varepsilon\mu} \nabla \text{div } \mathbf{A} = i\omega\mu\mu_0 \int d^3r' \underbrace{\left[G_0(\mathbf{r}, \mathbf{r}') \hat{\mathbf{I}} + \frac{1}{k^2} \nabla \text{div } G_0(\mathbf{r}, \mathbf{r}') \hat{\mathbf{I}} \right]}_{\hookrightarrow \hat{\mathbf{G}}(\mathbf{r}, \mathbf{r}')} \mathbf{j}(\mathbf{r}') \quad (8.14)$$

or

$$\hat{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0) = \left(\hat{\mathbf{I}} + \frac{1}{k^2} \nabla \otimes \nabla \right) G_0(\mathbf{r}, \mathbf{r}_0), \quad (8.15)$$

where $(\nabla \otimes \nabla)_{\alpha\beta} = \partial_\alpha \partial_\beta$, $\alpha, \beta = x, y, z$ is the tensor product of ∇ .

Example: Field of a point dipole in vacuum

Let us assume a monochromatic dipole, so from (8.3) we obtain

$$\mathbf{j} = -i\omega \mathbf{d}_0 \delta(\mathbf{r} - \mathbf{r}_0). \quad (8.16)$$

Substitution to (8.7) gives

$$\mathbf{E}(\mathbf{r}) = i \frac{1}{c\epsilon_0} k(-i\omega) \int d^3r' \hat{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \mathbf{d}_0 \delta(\mathbf{r} - \mathbf{r}_0) = \frac{k^2}{\epsilon_0} \hat{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0) \mathbf{d}_0 \quad (8.17)$$

or

$$E_\alpha(\mathbf{r}) = \left(\frac{4\pi}{\epsilon_0} \right) k^2 \hat{G}_{\alpha\beta}(\mathbf{r}, \mathbf{r}_0) d_{0\beta} \quad \text{units:} \quad \begin{pmatrix} \text{sgs} \\ \text{SI} \end{pmatrix}. \quad (8.18)$$

Remark: it is worth noting that if $\mathbf{d}_0 = (d_0, 0, 0)$ then the first column of $\hat{\mathbf{G}}$ corresponds to the field, induced by a dipole directed along the x -axis.

8.1.2 Near-, intermediate- and far-field parts of Green's function

Sometimes it is convenient to write (8.15) in different form:

$$\hat{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0) \stackrel{\mathbf{R}=\mathbf{r}-\mathbf{r}_0}{=} \hat{\mathbf{G}}(\mathbf{R}) = \frac{e^{ikR}}{4\pi R} \left[\left(1 + \frac{ikR - 1}{k^2 R^2} \right) \hat{\mathbf{I}} + \frac{3 - 3ikR - k^2 R^2}{k^2 R^2} \frac{\mathbf{R} \otimes \mathbf{R}}{R^2} \right]. \quad (8.19)$$

Though this expression looks bulky, it can be separated into several components, which have different distance dependence (containing different powers of kR)

$$\hat{\mathbf{G}}(\mathbf{R}) = \hat{\mathbf{G}}_{NF} + \hat{\mathbf{G}}_{IF} + \hat{\mathbf{G}}_{FF}, \quad (8.20)$$

where

$$(\text{near field} = \text{electrostatic}) \quad \hat{\mathbf{G}}_{NF} = \frac{e^{ikR}}{4\pi R} \frac{1}{k^2 R^2} \left(3 \frac{\mathbf{R} \otimes \mathbf{R}}{R^2} - \hat{\mathbf{I}} \right), \quad kR \ll 1$$

$$(\text{intermediate field}) \quad \hat{\mathbf{G}}_{IF} = \frac{e^{ikR}}{4\pi R} \frac{i}{kR} \left(\hat{\mathbf{I}} - 3 \frac{\mathbf{R} \otimes \mathbf{R}}{R^2} \right), \quad kR \sim 1$$

$$(\text{far-field} = \text{radiating}) \quad \hat{\mathbf{G}}_{FF} = \frac{e^{ikR}}{4\pi R} \left(\hat{\mathbf{I}} - \frac{\mathbf{R} \otimes \mathbf{R}}{R^2} \right), \quad kR \gg 1$$

The simplified electric field distribution illustrating the effect of different components in (8.20) is shown in Fig. 30.

8.2 Spontaneous relaxation and local density-of-state (IN A MIXED UNITS)

8.2.1 An expression for spontaneous decay

Consider an excited atom which interacts with vacuum (fig. 31). Relaxation may accrue to different channels which are related to a single photon in different modes.

According to Fermi's Golden Rule the transition speed γ is given by

$$\gamma = \frac{2\pi}{\hbar^2} \sum_f \left| \langle f | \hat{V} | i \rangle \right|^2 \delta(\omega_i - \omega_f), \quad [\gamma] = \left[\frac{1}{\text{time}} \right], \quad (8.21)$$

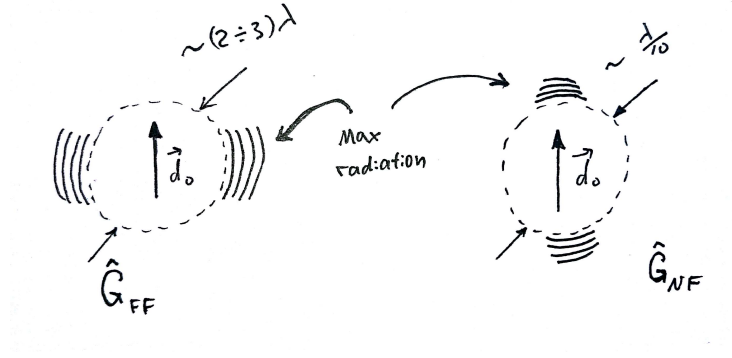


Figure 30: Intuitive picture of positioning of maximum radiation of a dipole

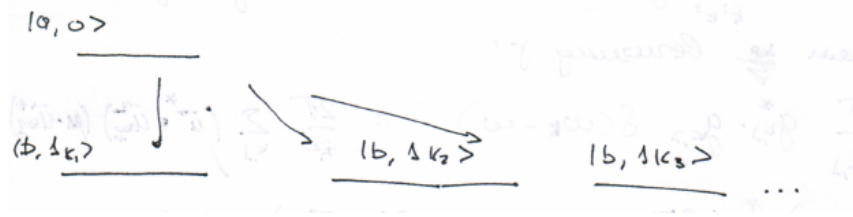


Figure 31: Transition from an initial state $|a, 0\rangle$ to a set of final states $|b, 1_{\{k\}}\rangle$. All the final states have the same energy. The states are products of atomic states and single-photon states.

where $|f\rangle = |b, 1_{\{k\}}\rangle$ and $|i\rangle = |a, 0\rangle$. In order to find it we need to calculate the matrix element. As we had above, perturbation operator is given by

$$\hat{V} = \hbar \sum_{\mathbf{k}, \lambda} g_{\mathbf{k}, \lambda} \hat{\sigma}_+ \hat{a}_{\mathbf{k}, \lambda} + \text{h.c.} \quad (8.22)$$

and field operator

$$\hat{\mathbf{E}} = \sum_{\mathbf{k}, \lambda} \varepsilon_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r} - i\omega_{\mathbf{k}}t} \mathbf{e}_{\mathbf{k}, \lambda} \hat{a}_{\mathbf{k}, \lambda} + \text{h.c.} = \hat{\mathbf{E}}^+ + \hat{\mathbf{E}}^-, \quad \varepsilon_{\mathbf{k}} = \sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}, \quad (8.23)$$

which is convenient to rewrite as

$$\hat{\mathbf{E}}^+ = \sum_{\mathbf{k}, \lambda} \varepsilon_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} \mathbf{e}_{\mathbf{k}, \lambda} \hat{a}_{\mathbf{k}, \lambda} e^{-i\omega_{\mathbf{k}}t} = \sum_{\mathbf{k}, \lambda} \mathbf{u}_{\mathbf{k}, \lambda}^+(\mathbf{r}) \hat{a}_{\mathbf{k}, \lambda} e^{-i\omega_{\mathbf{k}}t}, \quad (8.24)$$

where $\mathbf{u}_{\mathbf{k}}^+(\mathbf{r})$ — a vector of a mode which contains all the information about the system which is quantized. Omitting the computational details we have

$$|\langle f | \hat{V} | i \rangle|^2 = \hbar^2 |g_{\mathbf{k}, \lambda}|^2, \quad (8.25)$$

so

$$\gamma = \frac{2\pi}{\hbar^2} \sum_{\mathbf{k}, \lambda} \hbar^2 |g_{\mathbf{k}, \lambda}|^2 \delta(\omega_{\mathbf{k}} - \omega). \quad (8.26)$$

Using the fact that $g_{\mathbf{k},\lambda} = -\frac{\mathbf{d} \cdot \mathbf{u}_{\mathbf{k},\lambda}^+}{\hbar}$ we have

$$\begin{aligned} \gamma &= \frac{2\pi}{\hbar^2} \sum_{\mathbf{k},\lambda} (\mathbf{d} \cdot \mathbf{u}_{\mathbf{k},\lambda}^+)^* (\mathbf{d} \cdot \mathbf{u}_{\mathbf{k},\lambda}^+) \delta(\omega_{\mathbf{k}} - \omega) = \frac{2\pi}{\hbar^2} \sum_{\mathbf{k},\lambda} \mathbf{d}^* (\mathbf{u}_{\mathbf{k},\lambda}^- \otimes \mathbf{u}_{\mathbf{k},\lambda}^+) \mathbf{d} \delta(\omega_{\mathbf{k}} - \omega) = \\ &= \left/ \mathbf{u}_{\mathbf{k},\lambda}^+ = \varepsilon_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} \mathbf{e}_{\mathbf{k}\lambda} = \sqrt{2\pi\hbar\omega_{\mathbf{k}}} \cdot \frac{1}{\sqrt{V}} \mathbf{e}_{\mathbf{k}\lambda} e^{i\mathbf{k}\mathbf{r}} \stackrel{\text{def}}{=} \sqrt{2\pi\hbar\omega_{\mathbf{k}}} \mathbf{u}_{\mathbf{k}\lambda}^{0+} \right/ = \\ &= \frac{2\pi}{\hbar^2} \hbar\omega 2\pi |\mathbf{d}|^2 \sum_{\mathbf{k},\lambda} \mathbf{n}^* (\mathbf{u}_{\mathbf{k}\lambda}^{0+} \otimes \mathbf{u}_{\mathbf{k}\lambda}^{0-}) \mathbf{n} \delta(\omega_{\mathbf{k}} - \omega), \quad \mathbf{d} = |\mathbf{d}| \mathbf{n}. \end{aligned} \quad (8.27)$$

Now we introduce the *local density-of-state* which contains all the information about the system

$$\rho_{\mathbf{d}}(\mathbf{r}, \omega) \stackrel{\text{def}}{=} 3 \sum_{\mathbf{k},\lambda} \mathbf{n}^* (\mathbf{u}_{\mathbf{k}\lambda}^{0+} \otimes \mathbf{u}_{\mathbf{k}\lambda}^{0-}) \mathbf{n} \delta(\omega_{\mathbf{k}} - \omega), \quad (8.28)$$

then

$$\boxed{\gamma = \frac{4\pi^2\omega_0}{3\hbar} |\mathbf{d}|^2 \rho_{\mathbf{d}}, \quad |\mathbf{d}|^2 = |\langle a | e\mathbf{r} | b \rangle|^2.} \quad (8.29)$$

8.2.2 Spontaneous decay and Green's dyadics

After that need apply a Hilbert???Schmidt formula

$$\hat{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0, \omega) \sum_{\mathbf{k},\lambda} \frac{(\mathbf{u}_{\mathbf{k}\lambda}^{0+} \otimes \mathbf{u}_{\mathbf{k}\lambda}^{0-})}{\frac{\omega_{\mathbf{k}}^2}{c^2} - \frac{\omega^2}{c^2}}. \quad (8.30)$$

Using Sokhotski formula

$$\text{Im} \left\{ \frac{1}{\omega_{\mathbf{k}}^2 - \omega_0^2} \right\} = \frac{\pi}{2\omega_{\mathbf{k}}} \left(\delta(\omega_{\mathbf{k}} - \omega_0) + \underbrace{\delta(\omega_{\mathbf{k}} + \omega_0)}_{\hookrightarrow=0, \omega_{\mathbf{k}}>0} \right). \quad (8.31)$$

Let us consider the imaginary part of the dyadic Green's function

$$\text{Im} \left\{ \hat{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0, \omega) \right\} = \frac{\pi c^2}{2\omega} \sum_{\mathbf{k},\lambda} (\mathbf{u}_{\mathbf{k}\lambda}^{0+}(\mathbf{r}) \otimes \mathbf{u}_{\mathbf{k}\lambda}^{0-}(\mathbf{r}_0)) \delta(\omega - \omega_{\mathbf{k}}), \quad (8.32)$$

then the local density-of-state

$$\rho_{\mathbf{d}}(\mathbf{r}_0, \omega) = \frac{6\omega}{\pi c^2} \left(\mathbf{n}^* \cdot \text{Im} \left\{ \hat{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}_0, \omega) \right\} \cdot \mathbf{n} \right). \quad (8.33)$$

The main thing to notice is

$$\boxed{\gamma \sim \rho_{\mathbf{d}}(\mathbf{r}_0, \omega) \sim \text{Im} \left\{ \hat{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}_0, \omega) \right\}.} \quad (8.34)$$

Now let us calculate the spectral density of state

$$D(\omega) = \int d^3r_0 \rho_{\mathbf{d}}(\mathbf{r}_0, \omega), \quad (8.35)$$

then we find averaging over different orientations

$$\langle \rho_{\mathbf{d}} \rangle_{\text{orientations}} = \frac{6\omega}{\pi c^2} \frac{1}{3} \text{Im} \left\{ \text{Tr} \hat{\mathbf{G}} \right\} = \frac{\omega^2}{\pi^2 c^3}, \quad (8.36)$$

here we used

$$\text{Im} \left\{ \hat{G}_{\alpha\alpha}(\mathbf{r}_0, \mathbf{r}_0, \omega) \right\} = \frac{k}{6\pi}, \quad \alpha = x, y, z. \quad (8.37)$$

Remark: the real part has singularity

$$\lim_{\mathbf{r} \rightarrow \mathbf{r}_0} \text{Re} \left\{ \hat{G}_{\alpha\alpha}(\mathbf{r}, \mathbf{r}_0, \omega) \right\} = \infty, \quad \alpha = x, y, z. \quad (8.38)$$

Substitution to (8.29) gives us

$$(\text{vacuum}) \quad \gamma_0 = \frac{4|\mathbf{d}|^2\omega^3}{3\pi\hbar c^3} [\text{sgs}] = \frac{|\mathbf{d}|^2\omega^3}{3\hbar\varepsilon_0\pi c^3} [\text{SI}]. \quad (8.39)$$

For medium with $n = \sqrt{\varepsilon_m}$ but *with the same Green's function* we have

$$(\text{medium}) \quad \gamma_0 = \frac{4|\mathbf{d}|^2\omega^3}{3\pi\hbar c^3} n [\text{sgs}] = \frac{|\mathbf{d}|^2\omega^3}{3\hbar\varepsilon_0\pi c^3} n [\text{SI}]. \quad (8.40)$$

8.3 The Purcell factor

The Purcell factor is defined as

$$F_p \stackrel{\text{def}}{=} \frac{\gamma}{\gamma_0} = \frac{\text{Im} \left\{ \mathbf{n} \hat{\mathbf{G}}_{\text{new}} \mathbf{n} \right\}}{\text{Im} \left\{ \mathbf{n} \hat{\mathbf{G}} \mathbf{n} \right\}} = 1 + \frac{\text{Im} \left\{ \mathbf{n} \hat{\mathbf{G}}_{\text{ext}} \mathbf{n} \right\}}{\text{Im} \left\{ \mathbf{n} \hat{\mathbf{G}} \mathbf{n} \right\}} = 1 + \frac{\text{Im} \left\{ \mathbf{n} \hat{\mathbf{G}}_{\text{ext}} \mathbf{n} \right\}}{k/6\pi}, \quad (8.41)$$

where $\hat{\mathbf{G}}_{\text{new}} = \hat{\mathbf{G}} + \hat{\mathbf{G}}_{\text{ext}}$, γ — transition probability in a new media with its own $\hat{\mathbf{G}}_{\text{new}}$.

Now let us take a look from the classical standpoint. Radiation power is given by the integral

$$P = -\frac{1}{2} \text{Re} \int d^3r \mathbf{j}^* \mathbf{E}, \quad (8.42)$$

where dipole current and field is given by (8.16) and (8.17) respectively. After substitution we obtain

$$\begin{aligned} (\text{sgs}): \quad P &= \frac{\omega^2}{2} \text{Im} \int d^3r 4\pi k^2 \mathbf{d}^* \hat{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0, \omega) \mathbf{d} \delta(\mathbf{r} - \mathbf{r}_0) = \\ &= 2\pi k^2 \omega |\mathbf{d}| \text{Im} \left\{ \hat{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}_0, \omega) \right\} = 2\pi k^2 \omega |\mathbf{d}| \cdot \frac{k}{6\pi} \end{aligned} \quad (8.43)$$

and we've got the Rayleigh formula

$$P = \frac{k^3 \omega |\mathbf{d}|^2}{3} \sim \omega^4. \quad (8.44)$$

If a dipole is dieing down then its energy

$$W(t) = W(0) e^{-\gamma t} \quad (8.45)$$

and radiation power

$$P(t) = \frac{d}{dt} W(t) = -\gamma W(t). \quad (8.46)$$

At time $t \approx 0$ we obtain

$$\frac{P_{\text{medium}}(t)}{P_{\text{vacuum}}(t)} \approx \frac{\gamma}{\gamma_0} = F_p. \quad (8.47)$$

We have the same result but only using the classical approach. But one needs to remember that the reason of radiation has a quantum nature.

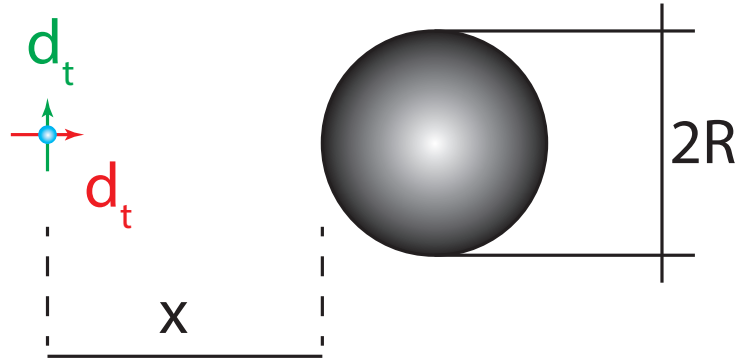


Figure 32: Atom with transverse and parallel dipole moment near a nanoparticle

Homework. Deadline: 29th of December

1. (6 pts) Calculate Purcell factor for an atom placed near a metal nanoparticle of radius R as a function of the distance to nanoparticle surface. Use dipole approximation considering the nanoparticle as a dipole with moment $\mathbf{d} = \alpha \mathbf{E}$, where $\mathbf{E}$ is the external electrical field calculated in the center of the nanoparticle. The polarizability of metal nanoparticle is $\alpha = R^3(\varepsilon - 1)/(\varepsilon + 2)$ (CGS units). Use Drude model for the dielectric permittivity $\varepsilon = 1 - \omega_p^2/(\omega(\omega + i\gamma))$, where the damping constant $\gamma = 0.1\omega_p$, and compute the distance dependence at the frequency of surface plasmon resonance $\omega = \omega_{SPR} = \omega_p/\sqrt{3}$. Consider two atom polarizations as shown in the figure.

9 Theory of relaxation of electromagnetic field. Heisenberg–Langevin method

9.1 In previous series

We have discussed:

1. Two-level system (e.g. atom) and one mode field. Result: Rabi oscillations.
2. Two-level system and continuum of modes (e.g. cavity). Result: spontaneous emission.
3. One-mode field in the cavity and continuum of modes. Result: let's find out!

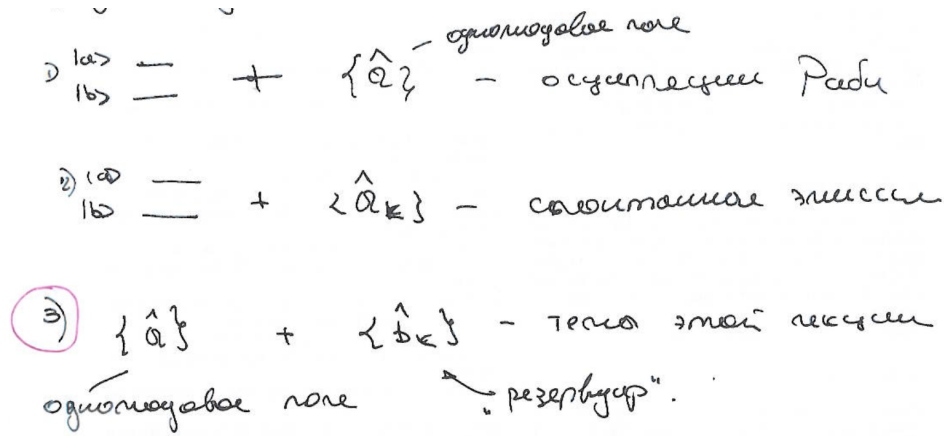


Figure 33: Different possible cases for analytical consideration.

9.2 the Heisenberg–Langevin equation

We are interested in dynamics of $\hat{b}(t)$ and $\hat{a}(t)$ in thermodynamics equilibrium at the temperature T . We consider modes which are fluctuate independently

$$\langle \hat{b}_k^\dagger(0) \hat{b}_{k'}(0) \rangle = \bar{n}_k \delta_{kk'}, \quad (9.1)$$

$$\langle \hat{b}_k(0) \rangle = \langle \hat{b}_k^\dagger(0) \rangle = 0, \quad (9.2)$$

$$\langle \hat{b}_k(0) \hat{b}_{k'}^\dagger(0) \rangle = (\bar{n}_k + 1) \delta_{kk'}, \quad (9.3)$$

$$\langle \hat{b}_k(0) \hat{b}_{k'}(0) \rangle = \langle \hat{b}_k^\dagger(0) \hat{b}_{k'}^\dagger(0) \rangle = 0. \quad (9.4)$$

Here and hereinafter the angle brackets represent the reservoir averages $\langle \dots \rangle \equiv \langle \dots \rangle_R$.

As the wave function is unknown we use the density matrix approach. Density matrix and distribution function in thermodynamics equilibrium is given by

$$\rho_R = \Pi \left(1 - e^{-\frac{\hbar\omega_k}{kT}} \right) e^{-\frac{\hbar\omega_k \hat{b}_k^\dagger \hat{b}_k}{kT}}, \quad \bar{n}_k = \frac{1}{e^{\frac{\hbar\omega_k}{kT}} - 1}. \quad (9.5)$$

The Hamiltonian of the system can be written as

$$\hat{\mathcal{H}} = \underbrace{\hbar\omega \hat{a}^\dagger \hat{a}}_{\text{resonator}} + \underbrace{\sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k}_{\text{cavity}} + \underbrace{\sum_k \hbar \left(g_k \hat{b}_k^\dagger \hat{a}_k + g_k^* \hat{a}_k^\dagger \hat{b}_k \right)}_{\hat{\mathcal{H}}_{\text{int}}}. \quad (9.6)$$

The explicit form of $g_{\mathbf{k}}$ is strongly depend on exact system. However we know the asymptotic behavior — if the resonator is ideal then $g_{\mathbf{k}} = 0$.

Time evolution is defined by

$$\hat{a}(t) = \frac{i}{\hbar} [\hat{\mathcal{H}}, \hat{a}], \quad \hat{b}(t) = \frac{i}{\hbar} [\hat{\mathcal{H}}, \hat{b}], \quad (9.7)$$

which gives

$$\begin{cases} \hat{a}(t) = -i\omega\hat{a}(t) - i \sum_{\mathbf{k}} g_{\mathbf{k}}^* \hat{b}_{\mathbf{k}}(t), \\ \hat{b}_{\mathbf{k}}(t) = -i\omega_{\mathbf{k}}\hat{b}_{\mathbf{k}}(t) - i g_{\mathbf{k}} \hat{a}(t), \end{cases} \quad (9.8a)$$

$$\quad (9.8b)$$

Important to notice that it is sensible to consider only the mean values of operators.

Now let us solve this system. it is easy to verify that the solution of (9.8b) is

$$\hat{b}_{\mathbf{k}}(t) = \underbrace{\hat{b}_{\mathbf{k}}(0)e^{-i\omega_{\mathbf{k}}t}}_{\text{reservoir modes evolution}} - \underbrace{i g_{\mathbf{k}} \int_0^t d\tau \hat{a}(\tau) e^{-i\omega_{\mathbf{k}}(t-\tau)}}_{\text{interaction between reservoir and oscillator}} \quad (9.9)$$

then we substitute this result in (9.8a) and obtain

$$\hat{a}(t) = -i\omega\hat{a}(t) - \hat{f}_a(t) + \sum_{\mathbf{k}} |g_{\mathbf{k}}|^2 \int_0^t d\tau \hat{a}(\tau) e^{-i\omega_{\mathbf{k}}(t-\tau)}, \quad (9.10)$$

where

$$\hat{f}_a(t) \stackrel{\text{def}}{=} i \sum_{\mathbf{k}} g_{\mathbf{k}}^* \hat{b}_{\mathbf{k}}(0) e^{-i\omega_{\mathbf{k}}t} \quad (9.11)$$

is a stochastic operator as the number of photons in a particular time moment is undefined. After that we introduce

$$\hat{\tilde{a}}(t) \stackrel{\text{def}}{=} \hat{a}(t) e^{i\omega t}, \quad \hat{\tilde{F}}_a(t) = \hat{f}_a(t) e^{i\omega t} \quad (9.12)$$

and obtain a *stochastic integro-differential equation*

$$\hat{\tilde{a}}(t) = \sum_{\mathbf{k}} |g_{\mathbf{k}}|^2 \int_0^t d\tau \hat{\tilde{a}} e^{i(\omega - \omega_{\mathbf{k}})(t-\tau)} + \hat{\tilde{F}}_a(t). \quad (9.13)$$

which is quite hard to solve, unless one was educated in the USSR.

To find a solution of (9.13) we, first of all, make transition from sum to the integral

$$\sum_{\mathbf{k}} |g_{\mathbf{k}}|^2 \rightarrow \frac{V}{\pi^2 c^3} \int_0^\infty d\omega_{\mathbf{k}} \omega_{\mathbf{k}}^2 |g_{\omega_{\mathbf{k}}}|^2. \quad (9.14)$$

Now let us consider only the integral in (9.13) which becomes

$$\begin{aligned}
& \frac{V}{\pi^2 c^3} \int_0^\infty d\omega_{\mathbf{k}} \omega_{\mathbf{k}}^2 |g_{\omega_{\mathbf{k}}}|^2 \int_0^t d\tau \hat{a}(\tau) e^{i(\omega - \omega_{\mathbf{k}})(t-\tau)} = \\
& = \left/ \omega_{\mathbf{k}}^2 |g_{\omega_{\mathbf{k}}}|^2 \text{ is a slow function} \right/ \approx \frac{V}{\pi^2 c^3} \int_0^t d\tau \hat{a}(\tau) \cdot \omega_{\mathbf{k}}^2 |g_{\omega_{\mathbf{k}}}|^2 \cdot \underbrace{\int_0^\infty d\omega_{\mathbf{k}} e^{i(\omega - \omega_{\mathbf{k}})(t-\tau)}}_{\hookrightarrow \approx 2\pi \delta(t-\tau)} = \\
& = \frac{V}{\pi^2 c^3} \omega_{\mathbf{k}}^2 \cdot 2\pi |g_{\omega_{\mathbf{k}}}|^2 \underbrace{\int_0^t d\tau \hat{a}(\tau) \delta(t-\tau)}_{\frac{1}{2} \hat{a}(t)} = \frac{\Gamma}{2} \hat{a}(t), \quad (9.15)
\end{aligned}$$

where we defined a relaxation constant Γ and density of states D :

$$\Gamma \stackrel{\text{def}}{=} D(\omega_{\mathbf{k}}) \cdot 2\pi |g_{\omega_{\mathbf{k}}}|^2, \quad D(\omega_{\mathbf{k}}) = \frac{V \omega_{\mathbf{k}}^2}{\pi^2 c^3} = \int d^3 r \rho(\mathbf{r}, \omega_{\mathbf{k}}). \quad (9.16)$$

So finally here is *the Heisenberg–Langevin equation*:

$$\boxed{\hat{\dot{a}}(t) = -\frac{\Gamma}{2} \hat{a}(t) + \hat{F}_a(t).} \quad (9.17)$$

Remark: we could obtain the same result if we would have considered the dynamics of atom's operator $\hat{\sigma}_z(t)$.

Example: Connection between dissipation and fluctuation

Let us put $\hat{F}_a(t) = 0$. It means that $\hat{a}(t) = \hat{a}(0)e^{-\frac{\Gamma}{2}t}$ which leads to

$$[\hat{a}(t), \hat{a}^\dagger(t)] = e^{-\Gamma t} \neq 1. \quad (9.18)$$

That is nonsense! It couldn't be, so we can not just put the stochastic term to zero. *If there is a dissipation then there have to be fluctuation in the system.*

Homework. Deadline: 29th of December

1. (4 pts) Consider two interacting cavities described by the Hamiltonian:

$$H = \hbar \omega a^\dagger a + \hbar \omega b^\dagger b + \hbar g(a^\dagger b + b^\dagger a)$$

Initially there were N_a and N_b photons in each resonator. Please, describe how the number of photon will change in time in each resonator.

Hint: use the equation of operator motion to calculate $\hat{n}_a$ and $\hat{n}_b$

9.3 Properties of the stochastic operator

Let us find out properties of the stochastic operator $\hat{F}_a(t)$. The mean values are

$$\langle \hat{F}_a(t) \rangle = \langle \hat{F}_a^\dagger(t) \rangle = 0. \quad (9.19)$$

Now let us find the correlator

$$\begin{aligned}
\langle \hat{F}_a^\dagger(t) \hat{F}_a(t') \rangle &= \sum_{\mathbf{k}\mathbf{k}'} g_{\mathbf{k}}^* g_{\mathbf{k}'} \cdot \underbrace{\langle \hat{b}_{\mathbf{k}}^\dagger(0) \hat{b}_{\mathbf{k}}(0) \rangle}_{\substack{\hookrightarrow = \bar{n}_{\mathbf{k}} \delta_{\mathbf{k}\mathbf{k}'} \\ \text{av. num. of ph. in } \mathbf{k}}} \cdot e^{i(\omega_{\mathbf{k}} - \omega)t - i(\omega_{\mathbf{k}'} - \omega)t'} = \\
&= \sum_{\mathbf{k}} |g_{\mathbf{k}}|^2 \bar{n}_{\mathbf{k}} e^{i(\omega_{\mathbf{k}} - \omega)(t - t')} = \int_0^\infty d\omega_{\mathbf{k}} D(\omega_{\mathbf{k}}) |g_{\mathbf{k}}|^2 \bar{n}_{\mathbf{k}}(\omega_{\mathbf{k}}) e^{i(\omega_{\mathbf{k}} - \omega)(t - t')} \approx \\
&\approx \underbrace{D(\omega_{\mathbf{k}}) |g_{\mathbf{k}}|^2 \bar{n}_{\mathbf{k}}(\omega_{\mathbf{k}})}_{\text{slow func}} \cdot 2\pi \delta(t - t'), \quad (9.20)
\end{aligned}$$

so we have

$$\langle \hat{F}_a^\dagger(t) \hat{F}_a(t') \rangle = \bar{n}_{\mathbf{k}} \Gamma \delta(t - t'). \quad (9.21)$$

Any stochastic function with δ -shaped correlator is called *white noise*. After integration over t' in (9.21) the dissipation rate can be written as

$$\boxed{\Gamma = \frac{1}{\bar{n}_{\mathbf{k}}(\omega_{\mathbf{k}})} \int_{-\infty}^{\infty} dt' \langle \hat{F}_a^\dagger(t) \hat{F}_a(t') \rangle}. \quad (9.22)$$

This relation takes its roots from the *fluctuation-dissipation theorem*.

Consider another averaging, that is the average of stochastic operator and an annihilation operator $\langle \hat{F}_a^\dagger(t) \hat{a}(t) \rangle$. From (9.17) we get

$$\hat{a}(t) = \hat{a}(0) e^{-\frac{\Gamma}{2}t} + \int_0^t d\tau \hat{F}_a(\tau) e^{-\frac{\Gamma}{2}(t-\tau)}, \quad (9.23)$$

so

$$\begin{aligned}
\langle \hat{F}_a^\dagger(t) \hat{a}(t) \rangle &= \underbrace{\langle \hat{F}_a^\dagger(t) \hat{a}(0) e^{-\frac{\Gamma}{2}t} \rangle}_{\hookrightarrow = 0, \text{ as } \langle \hat{F}_a \rangle = 0} + \int_0^t d\tau \langle \hat{F}_a^\dagger(t) \hat{F}_a(\tau) \rangle e^{-\frac{\Gamma}{2}(t-\tau)} \stackrel{(9.21)}{=} \frac{\bar{n}_{\mathbf{k}} \Gamma}{2}. \quad (9.24)
\end{aligned}$$

In a similar way it is easy to get the same result for $\langle \hat{a}(t) \hat{F}_a^\dagger(t) \rangle$, so we can write

$$\boxed{\langle \hat{F}_a^\dagger(t) \hat{a}(t) \rangle = \langle \hat{a}(t) \hat{F}_a^\dagger(t) \rangle = \frac{\bar{n}_{\mathbf{k}} \Gamma}{2}}. \quad (9.25)$$

These correlation functions will be employed to derive equations of motion for the field correlation functions.

9.4 Equation of motion for the field correlation functions. Wiener–Khinchine theorem

The mean time development of the field number operator is

$$\begin{aligned}
\frac{d}{dt} \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle &= \left\langle \frac{d\hat{a}^\dagger}{dt} \hat{a} \right\rangle + \left\langle \hat{a}^\dagger \frac{d\hat{a}}{dt} \right\rangle \stackrel{(9.17)}{=} -\frac{\Gamma}{2} \langle \hat{a}^\dagger \hat{a} \rangle + \underbrace{\langle \hat{F}_a^\dagger \hat{a} \rangle + \langle \hat{a} \hat{F}_a^\dagger \rangle}_{\hookrightarrow \stackrel{(9.25)}{=} \Gamma \bar{n}_{\mathbf{k}}} - \frac{\Gamma}{2} \langle \hat{a}^\dagger \hat{a} \rangle = \\
&= -\Gamma \langle \hat{a}^\dagger \hat{a} \rangle + \Gamma \bar{n}_{\mathbf{k}}. \quad (9.26)
\end{aligned}$$

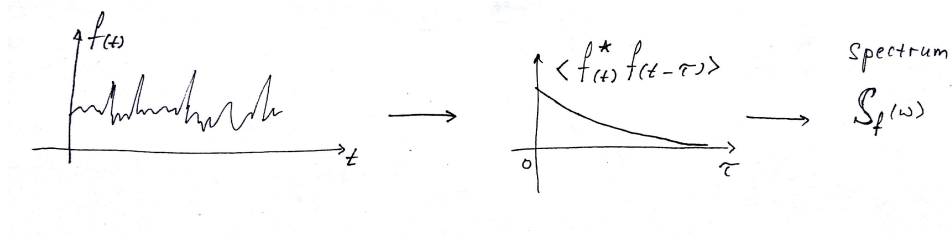


Figure 34: Wiener-Khintchine theorem in 40 seconds

In a similar manner, it can be shown that

$$\frac{d}{dt} \langle \hat{a}(t) \hat{a}^\dagger(t) \rangle = -\Gamma \langle \hat{a} \hat{a}^\dagger \rangle + \Gamma (\bar{n}_{\mathbf{k}} + 1). \quad (9.27)$$

To verify our results let us find the same commutator which was found in the example above. Shall we start with

$$\frac{d}{dt} \langle [\hat{a}(t), \hat{a}^\dagger(t)] \rangle = \Gamma \left(1 - \langle [\hat{a}(t), \hat{a}^\dagger(t)] \rangle \right). \quad (9.28)$$

We now what value should be at time $t = 0$, in other words we know the initial conditions for this differential equation, so letting $\zeta \stackrel{\text{def}}{=} \langle [\hat{a}(t), \hat{a}^\dagger(t)] \rangle$, we have

$$\begin{cases} \dot{\zeta} = \Gamma(1 - \zeta) \\ \zeta|_{t=0} = 1 \end{cases} \quad \rightarrow \quad \zeta = 1 \quad \rightarrow \quad \boxed{\langle [\hat{a}(t), \hat{a}^\dagger(t)] \rangle = 1}. \quad (9.29)$$

Which is correct in contrast to (9.18).

To define the spectrum we need to use *the Wiener-Khintchine theorem*. In short, this theorem allows to find spectrum using the knowledge of the correlation function (fig. 34) using relation

$$S_f(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \cdot \langle f^*(t) f(t + \tau) \rangle \quad (9.30)$$

Спектр написал, как было на лекциях, хотя в Скалли (ур-е 9.3.11) немного другое выражение, там $\int_0^\infty$ и берут действительную часть от интеграла.

In our case instead of $f(t)$ we have $\hat{a}(t) = \hat{a}(t)e^{-i\omega t}$ which gives

$$\langle \hat{a}^\dagger(t_0) \hat{a}(t_0 + \tau) \rangle = \langle \hat{a}^\dagger(t_0) \hat{a}(t_0 + \tau) \rangle e^{-i\omega\tau} = \langle n \rangle e^{-\frac{\Gamma\tau}{2}} e^{-i\omega\tau}, \quad (9.31)$$

where $\langle n \rangle \stackrel{\text{def}}{=} \langle \hat{a}^\dagger(t_0) \hat{a}(t_0) \rangle$ is the mean number of photons at the initial time t_0 . Then the spectrum

$$S(\nu) = \frac{1}{\pi} \langle n \rangle \int_{-\infty}^{\infty} d\tau e^{-i\nu\tau} \cdot e^{-\frac{\Gamma\tau}{2}} e^{-i\omega\tau} = \frac{1}{\pi} \langle n \rangle \frac{\Gamma/2}{(\nu - \omega)^2 + (\Gamma/2)^2}. \quad (9.32)$$

This is a Lorentzian distribution centered at $\nu = \omega$ with a half-width Γ (fig. 35). The quality factor is connected with our results and can be written as $Q = \frac{\omega}{\Gamma}$.

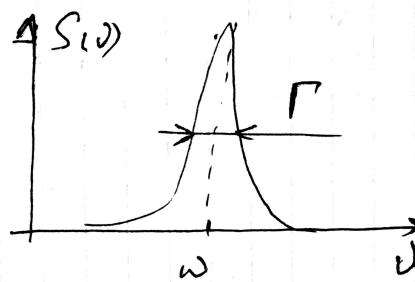


Figure 35: Spectrum has a Lorentzian distribution shape

10 Atom in a damped cavity

In the previous sections we have discussed different parts of the system (fig. 36) which we are going to combine in this section.

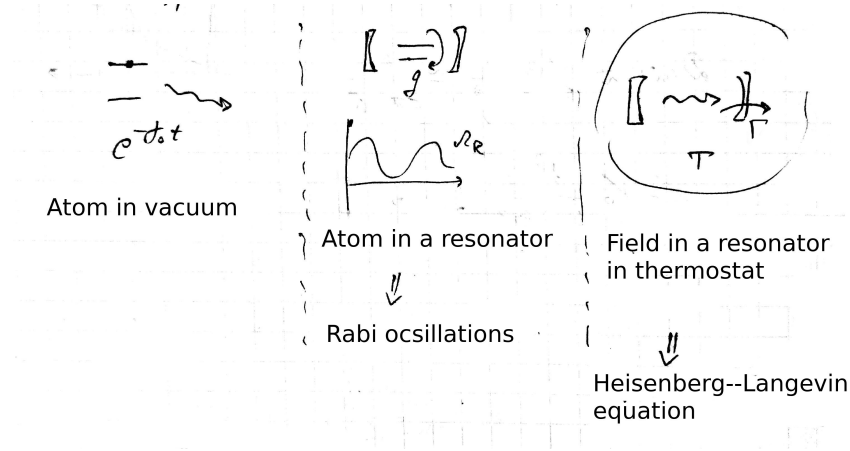


Figure 36: From left to right: atom in vacuum, atom in the cavity, field in a cavity in a thermostat.

Here we are going to study the evolution of a single two-level atom initially prepared in the upper level $|a\rangle$ of the transition resonant with the cavity mode (fig 37). In particular, it is seen that the spontaneous emission rate of the atom inside a resonant cavity is substantially enhanced over its free-space value (Purcell effect).

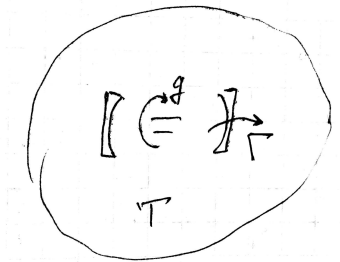


Figure 37: Atom in a cavity with losses. System has two coupling constants: g — atom and field coupling, Γ — cavity and reservoir coupling (transparency of a mirror)

10.1 The Purcell factor for a closed cavity

The decay rate γ can be written as

$$\gamma = 2\pi |g(\omega)|^2 \frac{D(\omega)}{V}, \quad (10.1)$$

where $D/V = \rho$ is the density of state. In vacuum it is $D_0(\omega) = \frac{\omega^2}{\pi^2 c^3}$ but in a cavity it can be approximated by the Lorentzian (fig) with resonant frequency ω_0

$$D(\omega) = \frac{1}{\pi} \frac{\omega_0/2Q}{(\omega - \omega_0)^2 + (\omega_0/2Q)^2}. \quad (10.2)$$

There two cases which is interesting to consider:

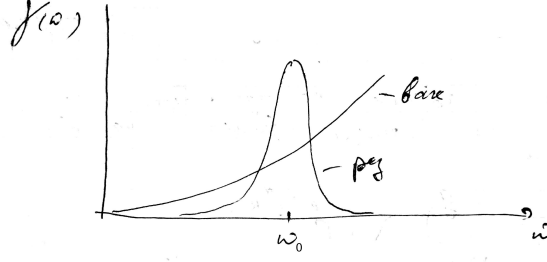


Figure 38: The decay rate in vacuum and in a cavity

Case 1. $\omega \approx \omega_0$ — transition and cavity frequencies are approximately equal. Then

$$D(\omega = \omega_0) \approx \frac{1}{\pi} \frac{2Q}{\omega_0}. \quad (10.3)$$

Substitution to (10.1) gives

$$\gamma = 2\pi |g(\omega)|^2 \frac{2Q}{\pi\omega} \frac{1}{V} = \underbrace{2\pi |g(\omega)|^2 \frac{\omega^2}{\pi^2 c^3}}_{\leftrightarrow \gamma_0} \cdot \frac{1}{V} \frac{\pi^2 c^3}{\omega^2} \cdot \frac{2Q}{\pi\omega} = \gamma_0 \frac{1}{(2\pi)^2} \frac{\lambda^3}{V} Q, \quad (10.4)$$

so the Purcell factor for a cavity is strongly depends on the λ^3/V and the quality factor Q :

$$F_P = \frac{\gamma}{\gamma_0} = \frac{1}{(2\pi)^2} \cdot \frac{\lambda^3}{V} Q. \quad (10.5)$$

Case 2. $|\omega_0 - \omega| \gg \Gamma = \omega_0/Q$. In this case we have

$$D(\omega) \approx \frac{1}{\pi} \frac{\Gamma}{2\omega^2} = \frac{1}{2\pi} \frac{1}{Q\omega}, \quad (10.6)$$

then

$$F_P = \frac{\gamma}{\gamma_0} = \frac{1}{(2\pi)^2} \frac{\lambda^3}{V} \cdot \frac{1}{Q}. \quad (10.7)$$

Usually $Q \gg 1$ and $\frac{\lambda^3}{V} \sim 1$, so far from the resonance $F_P \ll 1$.

10.2 Rigorous derivation of the atomic decay

If the approach is rigorous then we have to call Mr. Hamiltonian immediately

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_A + \hat{\mathcal{H}}_F + \hat{\mathcal{H}}_{AF} + \hat{\mathcal{H}}_R + \hat{\mathcal{H}}_{RF}, \quad (10.8)$$

where A stands for atom, F for field and R for reservoir. Summands of $\hat{\mathcal{H}}$ are the following

$$\hat{\mathcal{H}}_F = \hbar\omega\hat{n}, \quad \hat{\mathcal{H}}_A = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z, \quad \hat{\mathcal{H}}_R = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}\hat{n}_{\mathbf{k}}, \quad (10.9)$$

$$\hat{\mathcal{H}}_{AF} = \hbar g (\hat{\sigma}_+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_-), \quad \hat{\mathcal{H}}_{FR} = \hbar \sum_{\mathbf{k}} g_{\mathbf{k}} (\hat{a}^\dagger \hat{b}_{\mathbf{k}} + \hat{b}_{\mathbf{k}}^\dagger \hat{a}), \quad g, g_{\mathbf{k}} \in \mathbb{R}. \quad (10.10)$$

We are interested in time dependence of $\langle \hat{\sigma}_z \rangle$ and $\langle \hat{a}^\dagger \hat{a} \rangle$. Angle brackets represents the averaging over the reservoir and quantum mechanics averaging $\langle \dots \rangle \equiv \langle \langle \dots \rangle_{\text{R}} \rangle_{\text{QM}}$. For an any atom operator $\hat{O}_A$ we have a supporting relation

$$\frac{d}{dt} \langle (\hat{a}^\dagger)^m (\hat{a})^n \hat{O}_A \rangle = -\frac{i}{\hbar} \left[(\hat{a}^\dagger)^m (\hat{a})^n \hat{O}_A, \hat{\mathcal{H}}_A + \hat{\mathcal{H}}_F + \hat{\mathcal{H}}_{AF} \right] + \frac{d}{dt} \langle (\hat{a}^\dagger)^m (\hat{a})^n \rangle \hat{O}_A. \quad (10.11)$$

Similar to (9.26) it is not so hard to obtain the answer for a any positive integer powers

$$\begin{aligned} \frac{d}{dt} \langle (\hat{a}^\dagger)^m (\hat{a})^n \rangle &= \left[i\omega(m-n) - \frac{\Gamma}{2}(m+n) \right] \langle (\hat{a}^\dagger)^m (\hat{a})^n \rangle + \\ &+ \Gamma \cdot mn \cdot \bar{n}_R \langle (\hat{a}^\dagger)^{m-1} (\hat{a})^{n-1} \rangle. \end{aligned} \quad (10.12)$$

In the particular case, when $\hat{O}_A = \hat{1}$, we have

$$\frac{d}{dt} \langle \hat{a}^\dagger \hat{a} \rangle = -\frac{i}{\hbar} \langle [\hat{a}^\dagger \hat{a}, \hat{\mathcal{H}}_A + \hat{\mathcal{H}}_F + \hat{\mathcal{H}}_{AF}] \rangle - \Gamma \langle \hat{a}^\dagger \hat{a} \rangle + \Gamma \bar{n}_R. \quad (10.13)$$

As the commutator of $\hat{a}$ is trivial $[\hat{a}, \hat{a}^\dagger] = 1$, so

$$[\hat{a}^\dagger \hat{a}, g(\hat{\sigma}_+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_-)] = g\hat{a}^\dagger \hat{\sigma}_- - g\hat{\sigma}_+ \hat{a}, \quad (10.14)$$

then

$$\frac{d}{dt} \langle \hat{a}^\dagger \hat{a} \rangle = -\frac{i}{\hbar} g \underbrace{\langle \hat{a}^\dagger \hat{\sigma}_- - \hat{\sigma}_+ \hat{a} \rangle}_? - \Gamma \langle \hat{a}^\dagger \hat{a} \rangle + \Gamma \bar{n}_R, \quad (10.15)$$

where $\langle \hat{a}^\dagger \hat{\sigma}_- - \hat{\sigma}_+ \hat{a} \rangle$ is yet unknown.

Now let us take a look at sigma-z operator

$$\frac{d}{dt} \langle \hat{\sigma}_z \rangle = -\frac{i}{\hbar} \langle [\hat{\sigma}_z, g\hat{a}^\dagger \hat{\sigma}_- + g\hat{\sigma}_+ \hat{a}] \rangle = 2g \frac{i}{\hbar} \underbrace{\langle \hat{a}^\dagger \hat{\sigma}_- - \hat{\sigma}_+ \hat{a} \rangle}_?, \quad (10.16)$$

where we have got the same unknown averaging. To obtain last result the commutation relations for σ -matrices had been used:

$$[\hat{\sigma}_z, \hat{\sigma}_-] = -2\hat{\sigma}_-, \quad [\hat{\sigma}_z, \hat{\sigma}_+] = 2\hat{\sigma}_+. \quad (10.17)$$

To solve (10.15) and (10.16) we need to know the time dependence of that unknown Hermitean operator $\langle \hat{a}^\dagger \hat{\sigma}_- - \hat{\sigma}_+ \hat{a} \rangle$, which equation of motion involves the quantity $\langle \hat{a}^\dagger \hat{\sigma}_z \hat{a} \rangle$ and so on. In general, we get an infinite set of equations which may not be analytically solvable. The way out is the following — we need to cut the chain of equations at some step. Here the things we are going to suppose:

1. Put cavity at zero temperature reservoir ($\bar{n}_R = 0$).
2. Initially the atom is in the excited sate $|a\rangle$ and field inside the cavity is in the vacuum state $|0\rangle$.

In other words, we consider *a single photon approximation*. If the photon is only one then all summands which are proportional to $\hat{a}^2$ or $\hat{a}^{\dagger 2}$ are identically zero. As the result, under these conditions, we obtain the following system of equations:

$$\begin{cases} \frac{d}{dt} \langle \hat{a}^\dagger \hat{a} \rangle = g\chi(t) - \Gamma \langle \hat{a}^\dagger \hat{a} \rangle \\ \frac{d}{dt} \langle \hat{\sigma}_z \rangle = -2g\chi(t) \\ \frac{d}{dt} \chi(t) = g \langle \hat{\sigma}_z \rangle + 2g \cdot y(t) + g - \frac{\Gamma}{2}\chi(t) \\ \frac{d}{dt} y(t) = -g\chi(t) - \Gamma y(t) \end{cases}, \quad \begin{aligned} \chi(t) &\stackrel{\text{def}}{=} i \langle \hat{\sigma}_+ \hat{a} - \hat{a}^\dagger \hat{\sigma}_- \rangle, \\ y(t) &\stackrel{\text{def}}{=} \langle \hat{a}^\dagger \hat{\sigma}_z \hat{a} \rangle, \end{aligned} \quad (10.18)$$

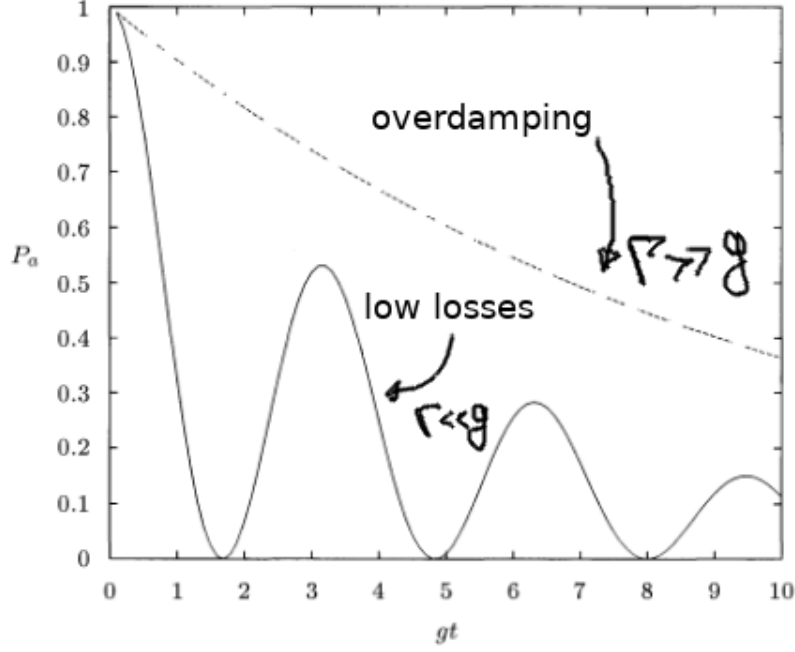


Figure 39: Probability P_a versus dimensionless time gt for the different limiting behavior regimes.

with the initial conditions

$$\chi|_{t=0} = 0, \quad y|_{t=0} = 0, \quad \langle \hat{a}^\dagger \hat{a} \rangle|_{t=0} = 0, \quad \langle \hat{\sigma}_z \rangle|_{t=0} = 1. \quad (10.19)$$

Using the single photon approximation is always simplifies the problem and make it look more "classical".

This system has two main parameters: g and Γ (see fig. 37). Let us consider two limiting behavior regimes:

Regime 1. $\Gamma \gg g \rightarrow \Gamma \gg \Omega_R$ or a *overdamping* regime. Here we have

$$\langle \hat{\sigma}_z \rangle = -1 + 2e^{-4g^2 \frac{t}{\Gamma}} = -1 + 2e^{-\gamma t}, \quad (10.20)$$

where

$$\gamma = \frac{4g^2}{\Gamma} = \frac{4g^2 Q}{\omega} = \underbrace{\frac{6\pi c^3 Q}{V\omega^2}}_{F_P} \cdot \frac{8|\mathbf{d}|^2 \omega^2}{\hbar \omega_0 c^3} \rightarrow \gamma \sim F_P. \quad (10.21)$$

The cavity increase the dissipation rate!

Regime 2. $\Gamma \ll g$ or a *low losses* regime. Here we can obtain that the atomic inversion $\langle \hat{\sigma}_z \rangle$ is

$$\langle \hat{\sigma}_z \rangle = -1 + e^{-\frac{\Gamma}{2}t} [1 + \cos(2gt)] \quad (10.22)$$

so the probability P_a take the simple form

$$P_a = \langle \hat{\sigma}_z \rangle + 1 = e^{-\frac{\Gamma}{2}t} [1 + \cos(2gt)] \quad (10.23)$$

Different regimes are represented on a fig. 39.

11 Casimir force and his close friends

This section is an overview of so-called *fluctuation-induced forces*. All effects here have a stochastic nature. Depending on the point of view this force may have different names: Casimir force, van der Waals force and Casimir–Polder force.

Example: A Frenchman of keen observation

In the XVIII century French navy men noticed one interesting thing. Two ships in a confused sea and with no-wind conditions being at a distance of 40 meters began to attract each other (Fig. 40). That happens because of the wave interference in the space between them. The calm sea between ships creates less pressure than the confused one outside.

Because of the same reason plastic islands originate.

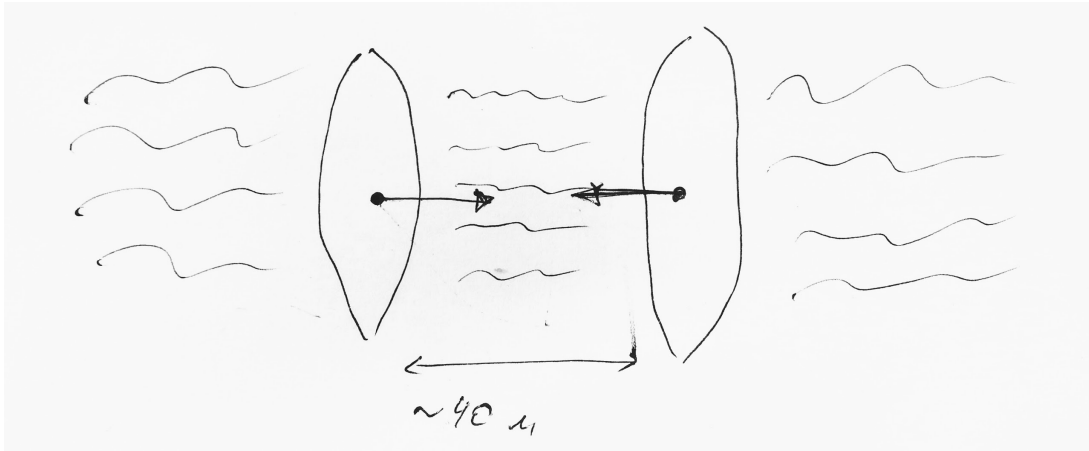


Figure 40: Two ships in a confused sea.

Fluctuating charges in a neutral body give rise to fluctuating electromagnetic fields which interact with the charges in other bodies. As a consequence, electromagnetic fields mediate between the charge fluctuations in separate bodies. The resulting charge correlations give rise to an electromagnetic force that is referred to as the dispersion force.

Example: A skillful Gecko

Geckos climb even the most slippery surfaces with ease and hang from glass using a single toe (Fig. 41). The secret behind this extraordinary climbing skill lies with millions of tiny keratin hairs on the surface of each foot. Although the dispersion force associated with each hair is miniscule, the millions of hairs collectively produce a powerful adhesive effect. The “gecko effect” is applied to the design of strongly adhesive tapes.



Figure 41: A nice Gecko climbs a vertical glass.

11.1 Casimir force between two perfectly conducting plates

Let us consider two parallel perfectly conducting metal plates in vacuum at the distance d . We choose z axis orthogonally to the plates (Fig. 42). Due to the boundary conditions waves have nodes where plates are. So wave vector k_z is quantized as

$$k_z = \frac{n\pi}{d}, \quad n \in \mathbb{Z}. \quad (11.1)$$

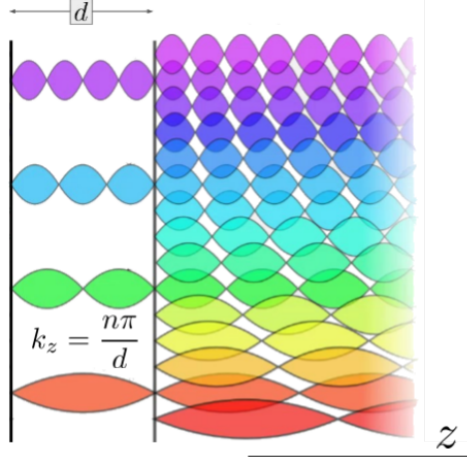


Figure 42: Perfectly conducting plates in vacuum. Wave vector k_z is quantized.

The main idea of calculating the force is the following. We expect that the force between two plates at the distance d is potential, so we could write

$$F_z(d) = - \left. \frac{dU}{dz} \right|_{z=d}. \quad (11.2)$$

It means that we need to find the potential energy of the plate interaction $U(d)$, which shows how much energy we need to expend to separate two plates from distance $z = d$ to $z \rightarrow \infty$, so

$$U(d) = E(d) - E(d \rightarrow \infty). \quad (11.3)$$

We need to remind that in the picture of secondary quantization Mr. Hamiltonian of photons looks as (??)

$$\hat{\mathcal{H}} = \sum_{\mathbf{k}, s} \hbar \omega_{\mathbf{k}} \left[\hat{a}_{\mathbf{k}, s}^\dagger \hat{a}_{\mathbf{k}, s} + \frac{1}{2} \right], \quad (11.4)$$

where $\omega_{\mathbf{k}} = c|\mathbf{k}|$. Our system has no real photons, so energy of the ground state $|0\rangle$ is

$$E = \langle 0 | \hat{\mathcal{H}} | 0 \rangle. \quad (11.5)$$

Further calculations depends on how many dimensions we live.

11.1.1 Case $D = 3$

In three dimensions we have

$$\omega_{\mathbf{k}} = c|\mathbf{k}| = c\sqrt{k_x^2 + k_y^2 + k_z^2} = \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{d}\right)^2}, \quad (11.6)$$

therefore interaction energy is

$$E = \sum_{\mathbf{k},s} \frac{\hbar\omega_{\mathbf{k}}}{2} = \hbar c A \int_{-\infty}^{\infty} \frac{dk_x dk_y}{(2\pi)^2} \sum_{n=0}^{\infty} \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{d}\right)^2}. \quad (11.7)$$

Here we used the fact that real photons may have two possible polarisations, so $\sum_s \rightarrow 2$. We also rewrote sum as $\sum_{\mathbf{k}} = \sum_{k_x, k_y} \sum_{k_z}$, where first sum was changed to the integral by the rule

$$\sum_{k_x, k_y} = A \int_{-\infty}^{\infty} \frac{dk_x dk_y}{(2\pi)^2}, \quad (11.8)$$

where A is the normalization area. $\sum_{k_z}$ should sum all possible values of k_z , due to the fact that k_z is quantised because of the boundary conditions it is a *countable infinity*, so

$$\sum_{k_z} g(k_z) = \sum_{n=0}^{\infty} g\left(\frac{n\pi}{d}\right), \quad (11.9)$$

where g is any function of k_z .

System has circular symmetry, so it is convenient to calculate integral in cylindrical coordinates where $\int dk_x dk_y = \int d\varphi dk_{\rho} k_{\rho} = 2\pi \int dk_{\rho} k_{\rho}$. It is also reasonable to look for the energy per area unit

$$\frac{E_s}{A} = \frac{\hbar c}{2\pi} \sum_{n=0}^{\infty} \int_0^{\infty} dk_{\rho} k_{\rho} \left[k_{\rho}^2 + \left(\frac{\pi n}{d}\right)^2 \right]^{\frac{1-s}{2}}, \quad (11.10)$$

where a new parameter s was introduced.

Remark: at this point we have made a bold move: we swapped $\sum_n$ and $\int dk_{\rho}$. By mathematical rigor we should have verified that both the sum and the integral are convergent. But we are physicist and we do need *right answers* other than a dense jungle of mathematical proof.

The main feat to calculate (11.10) is to introduce new parameter s and find E_s pretending that $\text{Re } s > 3$ and our life is fine and full of convergent integrals and sums. After that we just brassily say $E = \lim_{s \rightarrow 0} E_s$. Let us start

$$\begin{aligned} \frac{E_s}{A} &= \int x = k_{\rho}^2 / = \\ &= \frac{\hbar c}{2\pi} \sum_{n=0}^{\infty} \frac{1}{2} \int_0^{\infty} dx \left[x + \left(\frac{\pi n}{d}\right)^2 \right]^{\frac{1-s}{2}} = \frac{\hbar c}{4\pi} \sum_{n=0}^{\infty} \frac{2}{3-s} \left[x + \left(\frac{\pi n}{d}\right)^2 \right]^{\frac{3-s}{2}} \Big|_0^{\infty} = \\ &= -\frac{\hbar c}{2\pi} \frac{1}{3-s} \left\{ \sum_{n=0}^{\infty} \left(\frac{\pi n}{d}\right)^{3-s} - \sum_{n=0}^{\infty} \lim_{k_{\rho} \rightarrow \infty} k_{\rho}^{3-s} \right\}. \quad (11.11) \end{aligned}$$

One should not be afraid of the last term which is obviously divergent if we take limit $s \rightarrow 0$. As it was stated above we are searching for potential energy $U(d) = E(d) - E(d \rightarrow \infty)$. Considering $E(d \rightarrow \infty)$ in the same manner we get

$$E(d \rightarrow \infty) \sim \dots \sqrt{k_x^2 + k_y^2 + \lim_{d \rightarrow \infty} \left(\frac{\pi n}{d}\right)^2} = \dots \sqrt{k_{\rho}^2 + 0}. \quad (11.12)$$

This term precisely reduces second divergent term in (11.11).

Potential energy per area unit is

$$\frac{U(d)}{A} = -\lim_{s \rightarrow 0} \frac{\hbar c}{2\pi} \frac{1}{3-s} \frac{\pi^{3-s}}{d^{3-s}} \sum_{n=0}^{\infty} n^{3-s} = -\frac{\hbar c \pi^2}{6d^3} \sum_{n=0}^{\infty} n^3 = -\frac{\hbar c \pi^2}{720d^3}. \quad (11.13)$$

Here used regularized Riemann zeta-fuction

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad (11.14)$$

which in particular case gives $\zeta(-3) = \frac{1}{120}$. *Remark:* we will discuss such hardly intuitively understandable result later.

Finally, force acting on right metal plate per area unit is equal

$$\boxed{\frac{F_z(d)}{A} = \frac{\hbar c \pi^2}{240d^4}}. \quad (11.15)$$

11.1.2 Case $D = 1$ and philosophy about divergent sums

In one dimension case

$$\omega_{\mathbf{k}} = ck_z = \frac{n\pi}{d}, \quad (11.16)$$

so calculation are much easier, therefore it is easier to watch physics behind this effect.

At first, let us find energy

$$E = \langle 0 | \hat{\mathcal{H}} | 0 \rangle = \sum_{k_z, s} \frac{\hbar \omega_{\mathbf{k}}}{2} = \hbar c \frac{\pi}{d} \sum_{n=1}^{\infty} n = \left/ \zeta(-1) = -\frac{1}{12} \right/ = -\frac{\hbar c \pi}{12d}, \quad (11.17)$$

so force in one dimensional case is

$$\boxed{F_z = \frac{\hbar c \pi}{12d^2}}. \quad (11.18)$$

Here we again faced with a weird divergent sum which has a finite value. It not an artificial trick and it has physical reasons to do so. Infinite sum $\sum_{n=1}^{\infty} n$ represents a sum of all possible modes between plates, but all real metals have inductive impedance at high frequencies, so we may introduce a cutoff as

$$\sum_{n=1}^{\infty} n \rightarrow \sum_{n=1}^{\infty} n e^{-\varepsilon n}. \quad (11.19)$$

Now, considering $\varepsilon \ll 1$, using Taylor expansion we may write first terms

$$\sum_{n=1}^{\infty} n e^{-\varepsilon n} = \frac{e^{-\varepsilon}}{(1 - e^{-\varepsilon})^2} = \frac{1}{\varepsilon^2} \underbrace{-\frac{1}{12}}_{\text{important}} + \frac{\varepsilon^2}{240} + o(\varepsilon^4). \quad (11.20)$$

Neglecting all ε -dependent terms physically means that *we are looking for the difference between energy of two states: vacuum and vacuum with two plates*. In math it is called Dirichlet normalization.

11.2 Casimir–Polder force

In this part we are going to find fluctuation-induced force using point electric dipole approximation and Green's function (GF) formalism. Due to the fact that magnetic dipole and electric quadrupole interaction is usually the next order effect, point electric dipole approximation is enough for many cases.

Let us consider in average neutral objects with independent fluctuating field $\mathbf{E}^{\text{fl}}$ and dipole moment $\mathbf{p}^{\text{fl}}$. Now, we need to keep in mind that fluctuating field induces a dipole moment of the object with polarizability α may be written as

$$\mathbf{p}^{\text{in}}(\omega) = \alpha(\omega)\mathbf{E}^{\text{fl}}. \quad (11.21)$$

Moreover, any fluctuating dipole emits electromagnetic field. Induced field may be easily written using Green's function approach

$$\mathbf{E}^{\text{in}}(\mathbf{r}, \omega) = \frac{\omega^2}{c^2} \frac{1}{\varepsilon_0} \hat{\mathbf{G}}(\mathbf{r}, \mathbf{r}_0, \omega) \mathbf{p}^{\text{fl}}(\omega), \quad (11.22)$$

where $\mathbf{r}_0$ is the location of the fluctuating dipole.

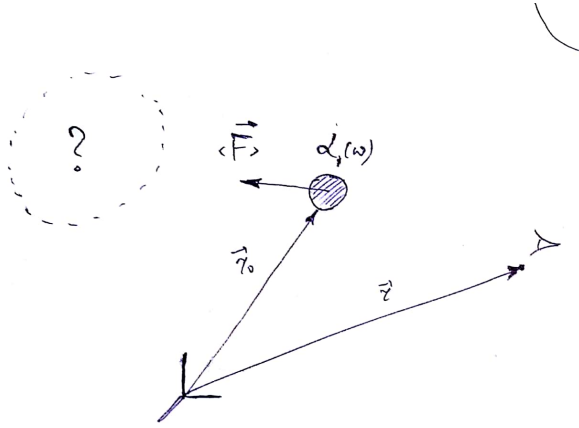


Figure 43: A small particle near unknown media, which is given by its Green's function.

From the electrodynamics an average electromagnetic force acting on a point dipole is given by

$$\langle \mathbf{F} \rangle = \left\langle \sum_{i=x,y,z} p_i \nabla E_i \right\rangle. \quad (11.23)$$

Both p and E may be induced or fluctuative, so $F \sim pE = (p^{\text{fl}} + p^{\text{in}})(E^{\text{fl}} + E^{\text{in}})$. Because of the fact that fluctuations are independent $\langle p^{\text{fl}} E^{\text{fl}} \rangle = 0$ and due to (11.21) and (11.22) $\langle p^{\text{in}} E^{\text{in}} \rangle \sim \langle E^{\text{fl}} p^{\text{fl}} \rangle = 0$. Other terms are proportional to the mean square of fluctuating values which are non-zero. Then (11.23) transforms into

$$\langle \mathbf{F} \rangle = \sum_i [\langle p_i^{\text{in}} \nabla E_i^{\text{fl}} \rangle + \langle p_i^{\text{fl}} \nabla E_i^{\text{in}} \rangle]. \quad (11.24)$$

Applying Fourier transform in respect that $\mathbf{E}^* = \mathbf{E}$ and with the help of (11.21) and (11.22) we obtain

$$\begin{aligned} \langle \mathbf{F} \rangle = & \sum_i \int d\omega d\omega' e^{i(\omega' - \omega)t} \alpha(\omega) \nabla \left\langle E_i^{\text{fl}}(\omega) E_i^{*\text{fl}}(\omega') \right\rangle + \\ & + \sum_i \int d\omega d\omega' e^{i(\omega' - \omega)t} \frac{\omega'^2}{c^2} \frac{1}{\varepsilon_0} \nabla \hat{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}_0) \left\langle p_i^{\text{fl}}(\omega) p_i^{*\text{fl}}(\omega') \right\rangle, \end{aligned} \quad (11.25)$$

where arrow shows which argument ∇ "hits". In this expression we don't know yet 3 things: (1) field correlator, (2) dipole correlator and (3) system Green's function.

Expressions for correlators may be obtained with the help of fluctuation-dissipation theorem (FDT). Dipole radiation is associated with the imaginary part of the polarizability¹. In liberal interpretation the FDT connects the fluctuating radioactive noise of the system with its losses. Without any additional details (see Novotny L.Principles of nano-optics, chapter 14):

$$\underbrace{\frac{2\pi i\omega}{kT} \langle p_j^{\text{fl}}(\omega) p_k^{*\text{fl}}(\omega') \rangle}_{\text{fluctuations}} = \underbrace{[\alpha_{ik}(\omega) - \alpha_{kj}^*]}_{\text{losses}} \delta(\omega - \omega'). \quad (11.26)$$

Moreover, there is a similar expression for field correlator

$$\langle E_j^{\text{fl}}(\mathbf{r}, \omega) E_k^{*\text{fl}}(\mathbf{r}', \omega') \rangle = \frac{\omega}{\pi c^2 \varepsilon_0} \delta(\omega - \omega') \left[\frac{\hbar \omega}{1 - e^{-\hbar \omega / kT}} \right] \text{Im } G_{jk}(\mathbf{r}, \mathbf{r}', \omega). \quad (11.27)$$

Combining this equations, for our system we get

$$\langle \mathbf{F}(\mathbf{r}_0) \rangle = \sum_i \int d\omega \frac{\omega}{\pi c^2 \varepsilon_0} \left[\frac{\hbar \omega}{1 - e^{-\hbar \omega / kT}} \right] \text{Im} \left\{ \alpha_i(\omega) \nabla G_{ii}(\mathbf{r}_0, \mathbf{r}_0, \omega) \right\}. \quad (11.28)$$

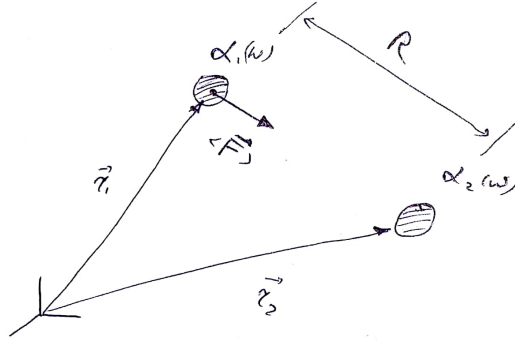


Figure 44: Definition of coordinates for the calculation of the dispersion force between two polarizable particles.

The only thing is left to find is the GF of the system. The electromagnetic dyadic GF connects a dipole moment with the total radiated field

$$\mathbf{E}(\mathbf{r}_2) = \frac{\omega^2}{c^2} \frac{1}{\varepsilon_0} \cdot \underbrace{\left(\text{system's GF} \right)}_{?} \cdot \mathbf{p}_1. \quad (11.29)$$

So, *all* the information about the system (reflections, self-action, self-consistent effects and so on) is hidden in the GF.

¹ Loss power

$$P_{\text{loss}} \sim \text{Re} \{ \mathbf{j}^* \mathbf{E} \} \sim \text{Re} \{ \alpha^* (-i\omega) |\mathbf{E}|^2 \} \sim |\mathbf{E}|^2 \text{Im } \alpha,$$

where we used

$$\mathbf{j} = \dot{\mathbf{p}} \delta(\mathbf{r} - \mathbf{r}_0) = -i\omega \delta(\mathbf{r} - \mathbf{r}_0) \mathbf{p} = -i\omega \delta(\mathbf{r} - \mathbf{r}_0) \alpha \mathbf{E}.$$

Now let us consider the simplest case: two point dipole with polarizabilities α_1 and α_2 . In other words, we replaced the unknown media in Fig. 43 by a another small particle. Total electric field at some point $\mathbf{r}$ may be written as

$$\mathbf{E}(\mathbf{r}) = \frac{k^2}{\varepsilon_0} \hat{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_1) \mathbf{p}_1 + \frac{k^2}{\varepsilon_0} \hat{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_2) \mathbf{p}_2, \quad (11.30)$$

where $\hat{\mathbf{G}}_0$ is the vacuum GF and $k^2 = \omega^2/c^2$. Now we want something that looks like (11.29). So we need to rewrite $\mathbf{p}_2$ in terms of $\mathbf{p}_1$. The dipole moment of the second particle is induced by the field from the first particle $\mathbf{E}_1(\mathbf{r}_2)$

$$\mathbf{p}_2 = \alpha_2 \mathbf{E}_1(\mathbf{r}_2) = \alpha_2 \frac{k^2}{\varepsilon_0} \hat{\mathbf{G}}_0(\mathbf{r}_2, \mathbf{r}_1) \mathbf{p}_1. \quad (11.31)$$

Now (11.30) transforms into

$$\mathbf{E}(\mathbf{r}) = \frac{k^2}{\varepsilon_0} \underbrace{\left(\hat{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_1) + \frac{k^2}{\varepsilon_0} \hat{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_2) \alpha_2 \hat{\mathbf{G}}_0(\mathbf{r}_2, \mathbf{r}_1) \right)}_{\text{system's GF}} \mathbf{p}_1. \quad (11.32)$$

In the more general case, for particles with non-isotropic polarizability we have

$$\hat{\mathbf{G}}^{\text{eff}}(\mathbf{r}, \mathbf{r}_1) = \hat{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_1) + \frac{k^2}{\varepsilon_0} \hat{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}_2) \hat{\alpha}_2 \hat{\mathbf{G}}_0(\mathbf{r}_2, \mathbf{r}_1). \quad (11.33)$$

After lots of calculations it appears that force is potential in a sense $U = - \int \langle F_R \rangle dR$ and potential may be written as

$$U(R) = - \frac{\hbar c}{16\pi^3 \varepsilon_0^2} \frac{1}{R^6} \int_0^\infty d\eta \alpha_1(i\eta) \alpha_2(i\eta) e^{-2\eta R} [3 + 6\eta R + 5(\eta R)^2 + 3(\eta R)^3 + (\eta R)^4]. \quad (11.34)$$

This is the Casimir–Polder potential. It is important to look at asymptotics:

- $R \rightarrow 0$:

$$U(R \rightarrow 0) \sim \frac{1}{R^6}. \quad (11.35)$$

This is the *van der Waals potential*.

- $R \rightarrow \infty$:

$$U(R \rightarrow \infty) \sim \frac{1}{R^7}. \quad (11.36)$$

This is the *Casimir potential*.

11.3 Orders of forces

Two small metal plates with side area $A = 1 \mu\text{m}$ and at the distance $d = 5 \text{ nm}$ in accordance to (11.15) attracts to each other with force $F \sim 0.1 \text{ mN}$, which is incredibly strong for a nanoworld. If we take in account the finite size of the plates, non-ideal conductivity and medium around then we get $F \sim 1 \text{ nN}$, which is still enough to crush a biomolecule! By the way, it is possible to measure forces up to several pN.

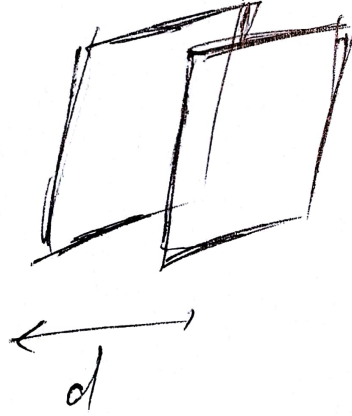


Figure 45: Two small metal plates with area $A = 1 \mu\text{m}^2$ at the distance $d = 5 \text{ nm}$ attracts to each other with the force $F \sim 10^{-9} \text{ N}$.

11.4 The latest advances

11.4.1 The dynamical Casimir effect

The dynamical Casimir effect is the production of particles and energy from an accelerated moving mirror. This reaction was predicted by certain numerical solutions to quantum mechanics equations made in the 1970s. In May 2011 an announcement was made by researchers at the Chalmers University of Technology, in Gothenburg, Sweden, of the detection of the dynamical Casimir effect. In their experiment, microwave photons were generated out of the vacuum in a superconducting microwave resonator.

Virtual photons between plates are seeds of real photons. The kinetic energy of moving mirrors transforms into radiative energy. This effect is possible only at very high frequencies, ideally plates should oscillate with an optical frequency. This can be obtained by modulating effective ε in the media between mirrors, so the optic length between them is also changing. Moreover, due to the conservation laws, new photons are entangled.

11.4.2 Quantum levitation or repulsive Casimir–Lifshitz forces

Let us consider a nanoparticle over a surface. With respect to some conditions attractive force may become repulsive. In such case this effect is called the *generalized Casimir effect* or *Casimir–Lifshitz effect*. See Fig. 47 for more details. This may be used as a building block for ultra slippery surfaces in the future.

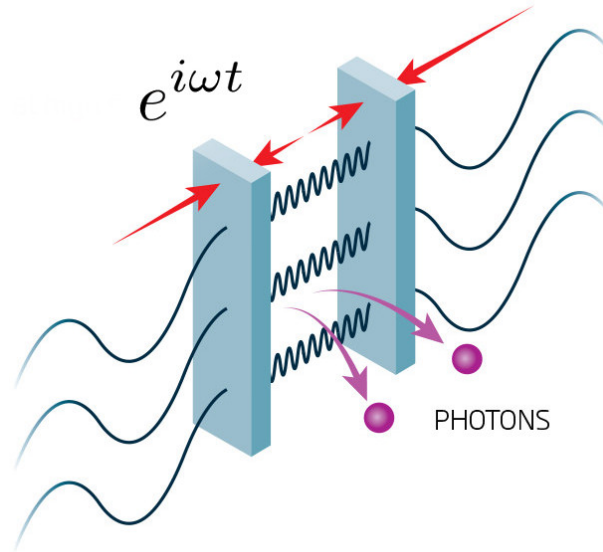
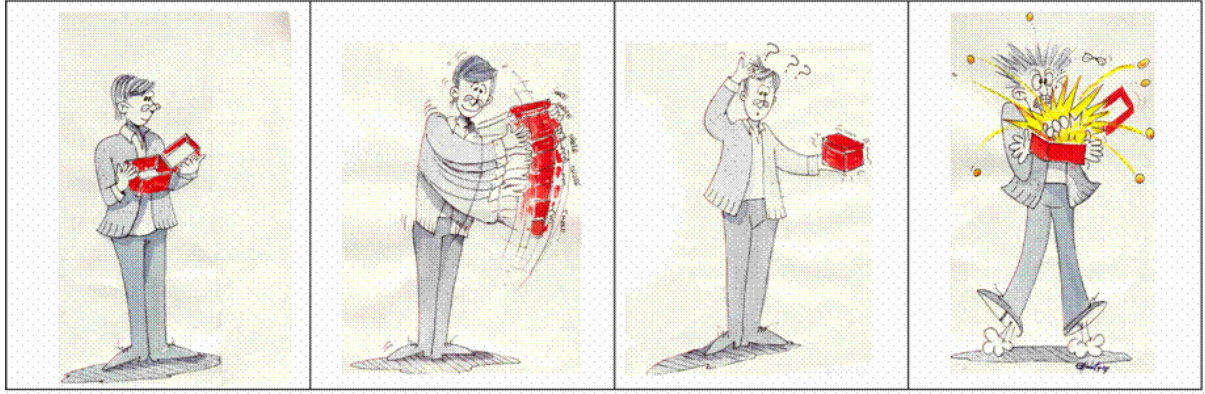
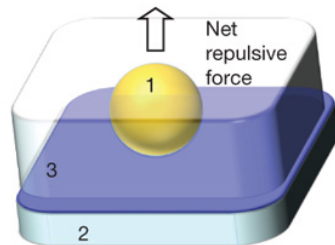


Figure 46: Metal plates brought together at very high speed (ideally at an optic frequency). As the result, due to conservations laws entangled photons are generated.



a



b

Figure 47: Repulsive quantum electrodynamical forces can exist for two materials separated by a fluid. (a) The interaction between material 1 and material 2 immersed in a fluid (material 3) is repulsive when $\varepsilon_1 > \varepsilon_3 > \varepsilon_2$. See details in *Nature* **457**, 170–173. (b) Cover of *Nature* issue about Casimir–Lifshitz force.